Two-Parameter Families of Strange Attractors

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Periodically driven two dimensional nonlinear oscillators can generate strange attractors that are periodic. These attractors are mapped in a locally 1-1 way to entire families of strange attractors that are indexed by a pair of relatively prime integers, (n_1, n_2), with $n_1 \geq 1$. The integers are introduced by imposing periodic boundary conditions on the entire strange attractor rather than individual trajectories in the attractor. The torsion and energy integrals for members of this two parameter family of locally identical strange attractors depend smoothly on these integers.

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Many strange attractors that exist in three dimensions can be embedded in a simple solid torus (solid tire tube). Such attractors include periodically driven two dimensional nonlinear oscillators as well as autonomous three dimensional dynamical systems. Each such strange attractor can be considered as the progenitor of an entire two-parameter family of strange attractors. Every member in this family has identical local properties: fractal dimensions and Lyapunov exponents. They differ in their global topological properties. The global properties are indexed by a pair of integers (n_1, n_2), with $n_1 \geq 1$ and n_2 relatively prime to n_1.

Simple nonlinear energy nonconserving equations have been used to describe the physical properties of electronic circuits, vibrating mechanical systems, and fluids [1–3]. The simplest nonlinear models assume the form of modified nonlinear oscillators: $\dot{X} = F_1(X, Y)$, $\dot{Y} = F_2(X, Y)$, typically with $F_1(X, Y) = Y$ and the nonlinearity contained in the second term, $F_2(X, Y)$. This is $\mu Y + X - X^3$, $\mu(1 - \beta X^2)Y - X^3$, and $\mu X + \nu Y + X^2 Y - X^3$ for the Duffing, van der Pol, and Takens-Bogdanov oscillators, respectively. These coupled nonlinear equations cannot exhibit chaotic behavior. However, if they are periodically driven they can exhibit chaotic behavior. The strange attractors that are generated by a periodic drive, of period T_d, are periodic and can have the same period. They exist in the phase space, the torus $D^2 \times S^1$. Here $D^2 \subset R^2$ is a disk and the relation between the geometric angle $\theta \in S^1$ and the time is $\theta/2\pi = t/T_d \mod 1$.

The equations of motion for a periodically forced oscillator often assume the form

$$\frac{d}{dt}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F_1(X,Y) \\ F_2(X,Y) \end{bmatrix} + \begin{bmatrix} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{bmatrix} \qquad (1)$$

with $\omega_d = 2\pi/T_d$. Usually $a_1 = 0$ or $a_2 = 0$ and $\phi_{1,2} = 0$ or $\pi/2$. In many instances the original forcing terms have inversion symmetry: $F_i(-X, -Y) = -F_i(X, Y)$. This is true for the Duffing, van der Pol, and Takens-Bogdanov oscillators. When this occurs the forced system, and any strange attractor of period T_d that it generates, has an internal order-two symmetry with generator $(X, Y, t) \to (-X, -Y, t + \frac{1}{2}T_d)$. This symmetry complicates the study of the strange attractor. This is because the stretching and folding that occurs during the first half period is repeated, inverted, during the second half period. If a description requiring three symbols would suffice to describe a trajectory during the first half period, up to 3^2 symbols would be required to describe a trajectory during the full period. Any trajectory through the strange attractor would be represented by a sequence of symbols drawn from an alphabet containing nine (or slightly fewer) letters [4].

For this reason it is useful to project the strange attractor onto one without the internal symmetry. This has been done by mapping it onto a van der Pol plane [5–7]. This is a rotating coordinate system in the disk, D^2, whose rotation is synchronized with the period of the driving term. The coordinates $(u(t), v(t), t)$ of the projected attractor are related to the original coordinates $(X(t), Y(t), t)$ by

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos\Omega t & -\sin\Omega t \\ \sin\Omega t & \cos\Omega t \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \qquad (2)$$

with $\Omega = \pm\omega_d$. Under this transformation $(u, v, t) \to (+u, +v, t + \frac{1}{2}T_d)$ [7]. The coordinate transformation (2) removes the internal symmetry and reduces the periodicity of the image attractor(s) to $T_1 = \frac{1}{2}T_d$. In the image attractor the geometric coordinate θ of the torus $D^2 \times S^1$ and the dynamical coordinate t of the attractor are related by $\theta/2\pi = t/\frac{1}{2}T_d = t/T_1$. The two choices $\Omega = \pm\omega_d$ define two counter-rotating van der Pol planes.

The dynamical system equations in the rotating coordinate system are

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = R\mathbf{F}(R^{-1}\mathbf{u}) + R\mathbf{t} + \Omega \begin{bmatrix} -v \\ +u \end{bmatrix} \qquad (3)$$

The first term on the right is the original forcing term in Eq. (1), without the periodic drive, as seen in a rotating coordinate system. The matrix $R = R(\Omega t) \in SO(2)$ is the rotation matrix in Eq. (2), and $R^{-1}(\Omega t) = R(-\Omega t)$ is its inverse. The second term in this equation is the periodic drive $\mathbf{t}$ at angular frequency ω_d, seen in the rotating coordinate system. The last term in this equation is the Coriolis term.

It is possible to map the coordinates $(X(t), Y(t))$ to other strange attractors using rotations that are harmonics of the drive: $\Omega = k\omega_d$, k integer. The choice

$k = 0$ reproduces the original attractor, the choices $k = \pm 1$ project onto the two counter-rotating van der Pol planes, and other integer choices of k project to yet additional strange attractors. All attractors are locally identical (locally diffeomorphic). Attractors constructed using distinct integers k are topologically inequivalent. This means that they cannot be smoothly deformed (isotoped) into each other, even if they are diffeomorphic [7, 8]. (For example, two identical belts, one cinched with a full twist, the other without, are diffeomorphic but topologically distinct, since one cannot be smoothly deformed into the other.) The internal symmetry for representations of the original strange attractor with integer index k is $(u, v, t) \to ((-1)^{k+1}u, (-1)^{k+1}v, t + \frac{1}{2}T_d)$. For even k all attractors have period T_d, possess the internal symmetry $(u, v, t) \to (-u, -v, t + \frac{1}{2}T_d)$, and are globally diffeomorphic with the original strange attractor. For k odd all have symmetry $(u, v, t) \to (u, v, t + \frac{1}{2}T_d)$ and thus have minimal period $\frac{1}{2}T_d$. These attractors (odd k) are $2 \to 1$ locally diffeomorphic images [9, 10] of the original attractor and globally diffeomorphic to, but topologically distinct from, each other as none can be smoothly deformed into any other.

Projections of the strange attractor generated by the following version of the van der Pol equations [6, 7]

$$\begin{aligned} \dot{X} &= bY + (c - dY^2)X \\ \dot{Y} &= -X + A\sin(\omega_d t) \end{aligned} \qquad (4)$$

with $(A, b, c, d, \omega_d, T_d) = (0.25, 0.7, 1.0, 10.0, \pi/2, 4.0)$ are shown in Fig. 1. This figure shows projections constructed using $k = -2, -1, 0, +1, +2$. The attractors constructed using harmonic rotations with $k = -2, 0, +2$ have period $T_d = 4.0$ and the projections with $k = -1, +1$ onto the van der Pol planes have minimal period $\frac{1}{2}T_d = 2.0$. They also show that the attractors become more tightly wound as $|k|$ increases, and change their direction of rotation as k passes through $k = 0$.

Two integrals express the relation between the rotation of the plane and the apparent rotation of the strange attractor as seen from that plane. These are the averaged angular momentum (or torsion) and energy integrals:

$$L(\Omega) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau (u\dot{v} - v\dot{u})dt = \langle u\ \dot{v} - v\ \dot{u}\rangle \qquad (5a)$$

$$K(\Omega) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \frac{1}{2}(\dot{u}^2 + \dot{v}^2)dt = \frac{1}{2}\langle \dot{u}^2 + \dot{v}^2\rangle \qquad (5b)$$

The values of the angular momentum and energy integrals for the rotating attractors ($\Omega = k\omega_d$) can easily be related to their values for the original ($k = 0$) strange attractor:

$$\begin{aligned} L(\Omega) &= L(0) + \Omega\langle R^2\rangle \\ K(\Omega) &= K(0) + \Omega L(0) + \frac{1}{2}\Omega^2\langle R^2\rangle \end{aligned} \qquad (6)$$

The average moment of inertia $\langle R^2\rangle = \lim_{\tau\to\infty}\int_0^\tau (u^2 + v^2)dt/\tau$ is independent of Ω. The values of these two integrals have been numerically computed ($\tau = 1000T_d$) for harmonic projections in the range $-10 \le k \le +10$. The angular momentum integral behaves linearly with Ω as $L(\Omega) = L(0) + \Omega\langle R^2\rangle$, with $\Omega = k\omega_d$. The energy integral behaves like Ω^2. More specifically, $K(\Omega) - K(0) = L(\Omega)^2/2\langle R^2\rangle - L(0)^2/2\langle R^2\rangle$. The energy integral has a

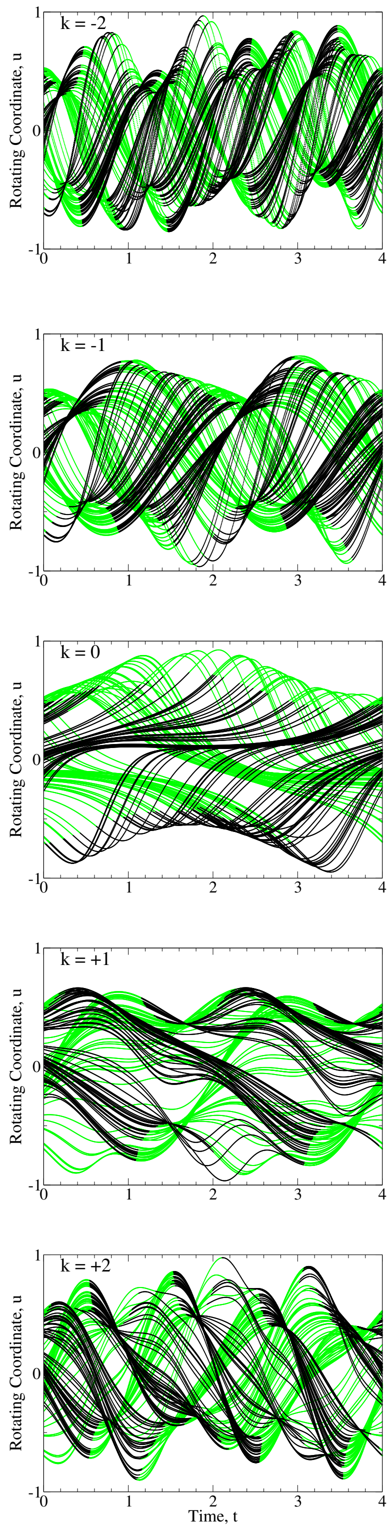

FIG. 1: (Color online) Strange attractors in the torus $D^2 \times S^1$ projected from the van der Pol strange attractor, with parameter values $(a, b, c, d, \omega_d, T_d) = (0.25, 0.7, 1.0, 10.0, \pi/2, 4.0)$ and rotation index k, $-2 \leq k \leq +2$.

minimum where the angular momentum integral has a zero crossing, at $k = 0.398$. The van der Pol image attractor with $k = +1$ is therefore the unique attractor in this set with minimal period and minimal energy. We therefore choose this attractor as the "universal image attractor."

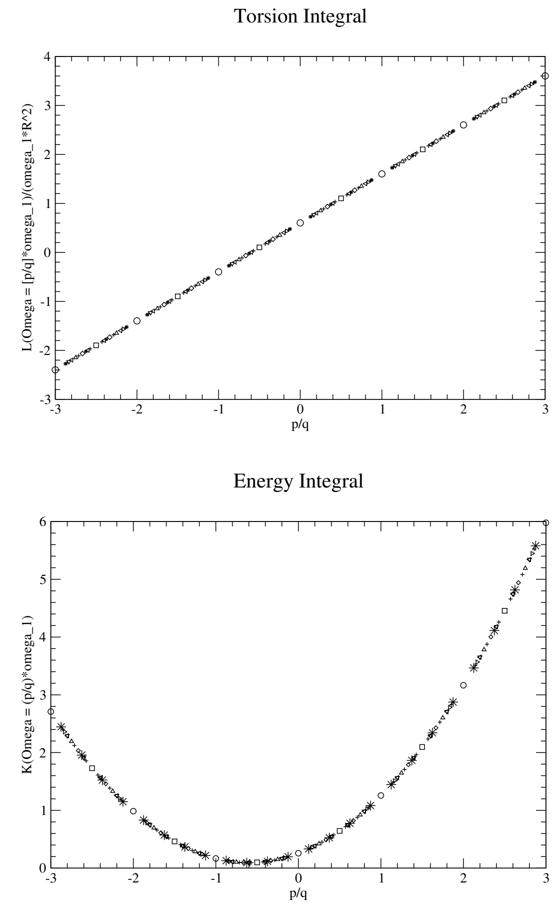

FIG. 2: Scaled torsion integral and energy integral for topologically inequivalent strange attractors with quantum numbers $(n_1, n_2) = (q, p)$, with $1 \leq n_1 \leq 8$ and $-3 \leq n_2/n_1 = p/q \leq +3$, q and p relatively prime.

The universal image attractor can be lifted to n_1-fold covers [9, 10] by using a rotating coordinate transformation as in Eq. 2, with $\Omega = (n_2/n_1)\omega_1$, acting on its coordinates (u, v). The lifted attractor satisfies periodic boundary conditions with $T_{(n_1,n_2)} = n_1 T_1$. During this time the rotation operation $R(\Omega t) \in SO(2)$ executes n_2 full rotations. The strange attractor constructed using this coordinate transformation is labeled (n_1, n_2). The lift $(2, -1)$ is the original van der Pol attractor. The torsion and energy integrals for these attractors depend smoothly on $\Omega = (n_2/n_1)\omega_1$. The integrals $L(\Omega)$ and $K(\Omega)$ have been estimated numerically for $1 \leq n_1 \leq 8$ and $-3 \leq n_2/n_1 \leq +3$ for $\tau = 2000T_1$ and are shown in Fig. 2. The scaled angular momentum, $L(\Omega)/(\omega_1 \langle R^2 \rangle)$, has been presented in Fig. 2 since the ratio is linear in n_2/n_1 with slope $+1$. The zero crossing of $L(\Omega)$ and minimum of $K(\Omega)$ are shifted by -1, since they are computed using the universal image attractor.

The two relatively prime integers (n_1, n_2) that label topologically inequivalent (nonisotopic) members in a family of strange attractors have been introduced in such a way that periodic boundary conditions are satisfied. This has been carried out by applying periodic boundary conditions to the entire strange attractor (Ω limit set) rather than individual trajectories in the attractor. In this sense the relatively prime integers (n_1, n_2) are like quantum numbers. This matching of boundary conditions, or quantization, is done along the two generators of the homotopy group [8] of the boundary of the solid torus $D^2 \times S^1$ containing the universal image strange attractor. The boundary is the torus $T^2 = S^1 \times S^1 \simeq \theta \times \phi$. The first space $S^1 = S^1_\theta$ describes the dynamics, with the relation $\theta/2\pi = t/(n_1 T_1)$ identifying the quantization condition along the longitude. The second space $S^1 = S^1_\phi = SO(2)_\phi$ describes the θ-dependent choice of rotating coordinate system in R^2, with the relation $\phi = (n_2/n_1)\theta$ identifying the quantization condition along the meridian. These quantization conditions are not restricted to periodically driven dynamical systems in $D^2 \times S^1$: they can be applied to any strange attractor that is contained in a bounding torus of genus one [11]. This has been done for the Rössler attractor [12].

All strange attractors (n_1, n_2) in this class are locally diffeomorphic. As a result, their spectra of fractal dimensions, Lyapunov exponents, and Lyapunov dimensions are identical. They are all topologically distinct, and can be distinguished one from the other by the spectrum and organization of their periodic orbits (linking numbers, relative rotation rates), their covering indices, their global topology (all integer or rational fraction quantities) and to some extent their torsion and energy integrals, which are not generally rational.

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