Chapter 3

Fractals

3.1 Examples of Fractals

A fractal is a geometric object which is self-similar, with structure at all levels of magnification. Rather than try to tighten down on this definition, it is more useful to generate some examples.

Example 1: In Fig. 3.1(a) we show an interval of length 1. In going from (a) to (b) we remove the middle half of this interval. This leaves two intervals, each of equal length $\frac{1}{4}$. This first step is the **generating step**. The second step, from (b) to (c), is a repeated application of the generating step. We remove the middle half of each of the two subintervals. This leaves $4 = 2^2$ intervals, all of equal length $\frac{1}{16} = (\frac{1}{4})^2$. We continue in the obvious way. At the n^{th} step we have 2^n intervals, each of length $(\frac{1}{4})^n$. This process continues forever.

3.2 Fractal Dimension

A convenient way to define the dimension of a geometric object is to cover it with boxes whose edge length is ϵ (*i.e.*, small). In Fig. 3.2 we show how this process works for some familiar geometric objects: two points, a smooth curve, and a simple area. In these three examples, the number of boxes, $N(\epsilon)$, required to cover the geometric objects behaves like:

Geometric Object	$N(\epsilon)$
Points	$P \sim K/\epsilon^0$
Smooth Curves	$C \sim K/\epsilon^1$
Simple Areas	$A \sim K/\epsilon^2$

where K is an unimportant constant. The number of boxes required to cover the geometric object behaves like ϵ^{-d}, where d is the dimension of the object. We can turn this observation around, and use this type of computation to *define* the dimension of peculiar objects.

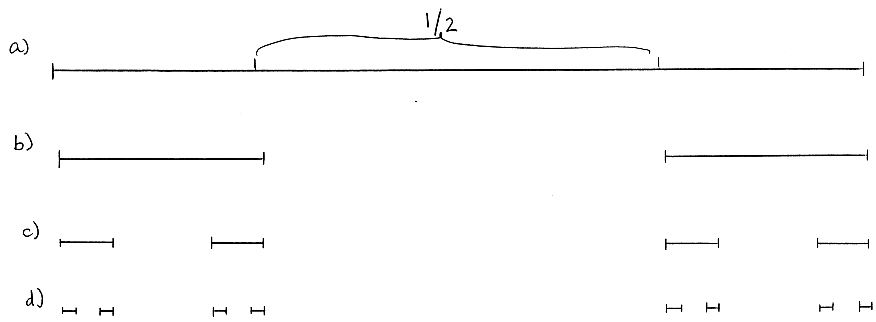

Figure 3.1: A middle half fractal is constructed by repeated application of the first, generating step. The middle half of the interval of length one (a) is removed (b). At each succeeding step, the middle half of each interval is removed. This continues forever.

3.2.1 Definition of Dimension (Box Counting)

Definition: We define the dimension, d, of a geometric in terms of ϵ and $N(\epsilon)$ as follows:

$$d = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)} \tag{3.1}$$

Example 1 (Continued): At the n^{th} step of the generation process of the middle half fractal, there are 2^n boxes, each of length $(\frac{1}{4})^n$. The fractal dimension is therefore

$$d = \lim_{n \to \infty} \frac{\log(2^n)}{\log(1/\frac{1}{4})^n} = \frac{\log 2}{\log 4} = \frac{1}{2}$$

3.2.2 Dimension of the Middle 1/p Fractal

Example 2: We can generalize this to $\frac{1}{p}$ fractals. These are fractals in which the middle $\frac{1}{p}$ of the interval is removed in the generating step. Each interval obtained during the generating step has length $\epsilon = \frac{p-1}{2p}$. Then

$$d = \frac{\log 2}{\log(\frac{2p}{p-1})}$$

For $p = 2, 3, 4, \cdots$ these dimensions are

p	Dimension
2	$\frac{1}{2}$
3	$\log(2)/\log(3)$
4	$\log(2)/\log(8/3)$
5	$\log(2)/\log(5/2)$

We plot the fractal dimension, d, as a function of $f = 1/p$ in Fig. 3.3.

3.2.3 Direct Product Spaces, Direct Sum Dimensions

Fractals in higher dimensional spaces can be built up systematically as direct products of fractals in lower dimensional spaces. If a fractal is a direct product of two fractals with dimensions d_1 and d_2, then its dimension is the (direct) sum of the dimensions of the two fractals:

$$d = d_1 + d_2$$

As an example, a fractal in the plane can be constructed as the direct product of the middle half fractal along each of the axes. The dimension of this direct product fractal is then

$$d = \frac{1}{2} + \frac{1}{2} = 1$$

It is clear from this example that fractals can have integer dimension.

3.3 Two Scale Fractals

3.3.1 Construction

Another way to build up fractals is shown in Fig. 3.4. In the generating step, an interval of length 1 is reproduced twice, once reduced by the scale factor λ_1, the other time reduced by the scale factor λ_2. These reduced intervals are shown on the left and right in Fig. 3.4(b). The process is repeated in the second generation. This produces four subintervals, of lengths $\lambda_1^2,\ \lambda_1\lambda_2,\ \ \lambda_2\lambda_1,\ \lambda_2^2$, proceeding from left to right. In the third generation the distribution is $\lambda_1^3,\ 3\lambda_1^2\lambda_2,\ 3\lambda_1\lambda_2^2,\ \lambda_2^3$. You can see the binomial distribution of lengths emerging from this process, which of course continues forever, as before.

3.3.2 Dimension

The dimension of this two scale fractal can be computed as follows. Assume that at level k, $N_k(\epsilon)$ boxes of length ϵ are required to cover the 2^k intervals. At the next level $k + 1$, the structure on the left is a scaled down version of the entire structure at level k. Therefore the number of boxes of length ϵ required

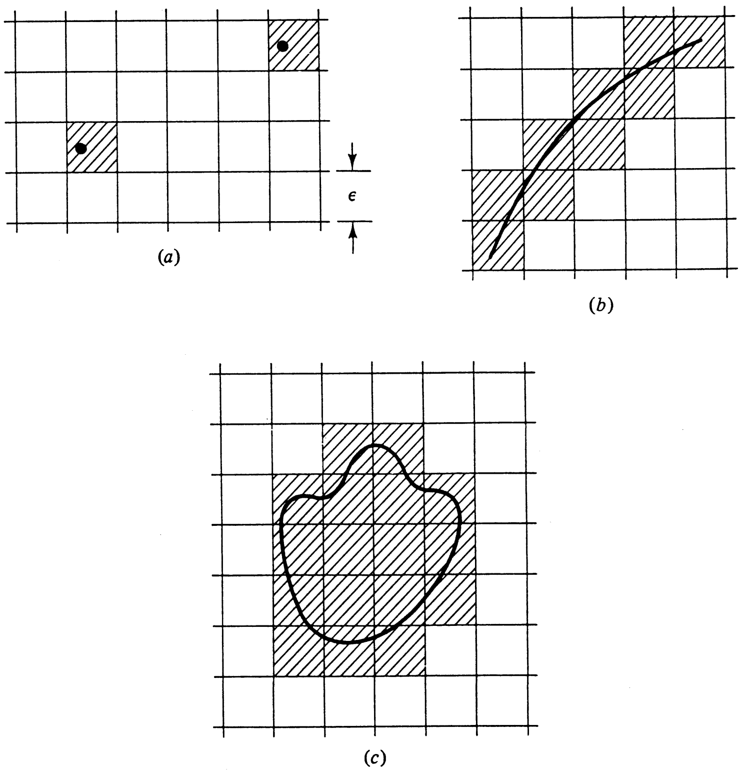

Figure 3.2: (Ott, p. 70)(a) Two boxes cover two points, no matter how small the boxes are. (b) The number of boxes required to cover a smooth curve is proportional to the length of the curve, and inversely proportional to the box size, that is, $N(\epsilon) \sim 1/\epsilon$. (c) The number of boxes required to cover the area behaves like $N(\epsilon) \sim 1/\epsilon^2$.

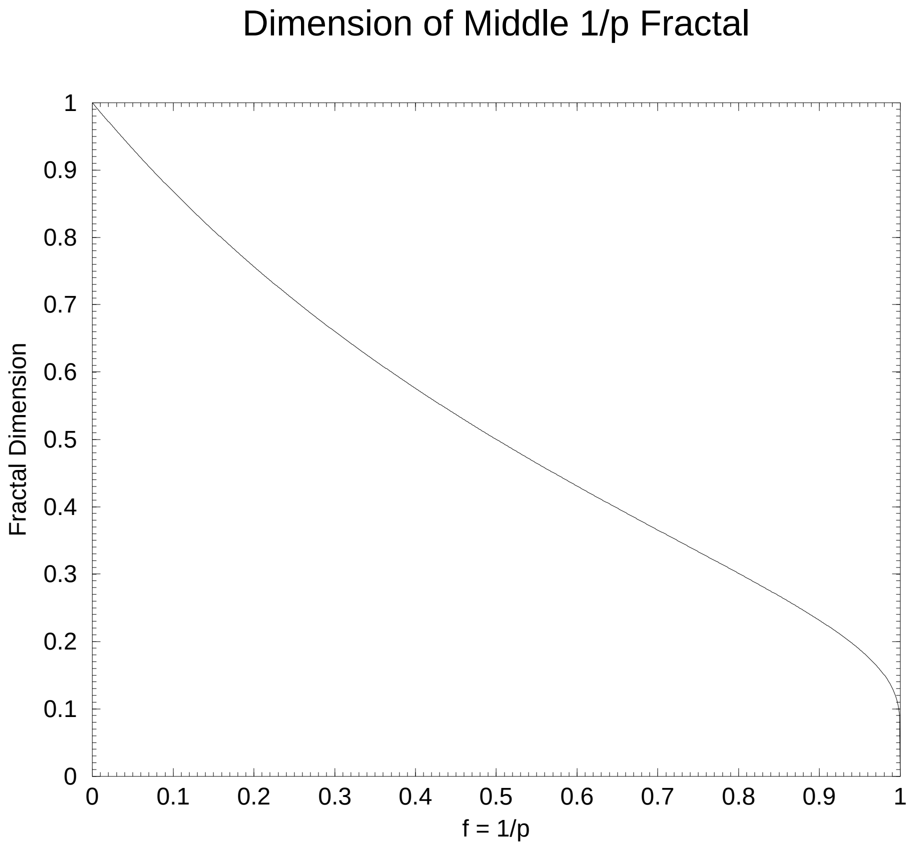

Figure 3.3: The dimension of a middle $1/p$ fractal is plotted as a function of $f = 1/p$.

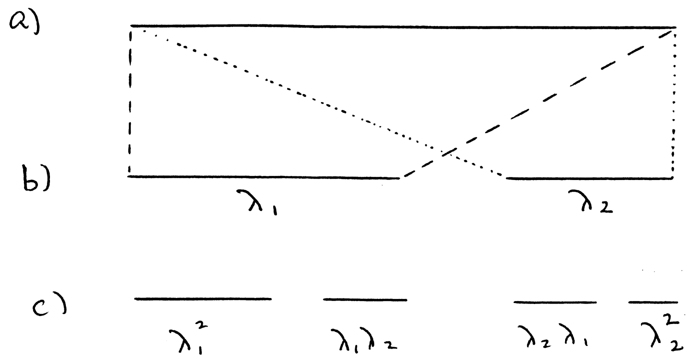

Figure 3.4: Construction of a two scale fractal proceeds as shown. Each of the two subintervals in the generating stage $(a) \rightarrow (b)$ is a replica of the original, reduced in scale by the scale factors λ_1 and λ_2. If λ_1 is negative, $-1 < \lambda_1 < 0$, the orientation of an interval is reversed when scaled down by λ_1.

to cover the left half of the structure at level $k+1$ is equal to the number of larger boxes (of length ϵ/λ_1) required to cover the structure at level k:

$$N_{(k+1)_{left}}(\epsilon) = N_k(\epsilon/\lambda_1)$$

A similar argument holds for the half on the right at level $k+1$. Thus we have

$$N_{k+1}(\epsilon) = N_{(k+1)_{left}}(\epsilon) + N_{(k+1)_{right}}(\epsilon) = N_k(\epsilon/\lambda_1) + N_k(\epsilon/\lambda_2)$$

If we assume, as usual, that $N(\epsilon) \sim K\epsilon^{-d}$, then $K\epsilon^{-d} = K(\epsilon/\lambda_1)^{-d} + K(\epsilon/\lambda_2)^{-d}$ leads directly to a simple expression defining the fractal dimension d:

$$\lambda_1^d + \lambda_2^d = 1$$

Fractals obtained from three or more scaling transformations in the generating step have dimensions determined by similar expressions.

3.3.3 Feigenbaum Fractal

The Feigenbaum attractor is the fractal which exists at the accumulation point of the period doubling cascade. Figure 3.5(a) shows the locations of orbits of periods 1, 2, 4, 8, $\cdots$. The locations suggest that a scaling exists. This scaling is reinforced in Fig. 3-5(b), which shows the locations of points in the orbits of 2^n. These occur alternately in the left and the right halves of the return plot, and seem to obey scaling $1/\alpha^2$ on the left and $1/\alpha$ on the right.

We now describe how to view the Feigenbaum attractor as a two scale fractal. Begin by connecting the two points in the period two orbit by a line. Next, connect the four points of the period four attractor by two lines. Continue on in this way, connecting the 2^n points in the period 2^n orbit by half the number of lines. This is shown in Fig. 3.6. The transformation from level n to the next lower level $n+1$ is then governed by the two scalings $\lambda_1 = 1/\alpha$ (since α is negative, this reverses orientation, but this is not important for the computation to follow) and $\lambda_2 = 1/\alpha^2$. The dimension of this attractor is therefore determined by

$$(1/|\alpha|)^d + (1/\alpha^2)^d = 1$$

The dimension d can easily be computed by the 'divide and conquer' strategy. It lies between 0 and 1. At $d = 1$ the left hand side is $1+1 = 2$. At $d = 1$ the left hand side is $\lambda_1 + \lambda_2 < 1$. This means that there is a zero crossing of $\lambda_1^d + \lambda_2^d - 1$ between 0 and 1. Evaluate in the middle, move the limits, and evaluate in the middle again. Continue for as long as you wish (no longer than the bit length of words in your computer). The snippet of FORTRAN code in Fig. 3.7 finds the value of d to 25 bits: $d = 0.52450\ 83040\ \cdots$.

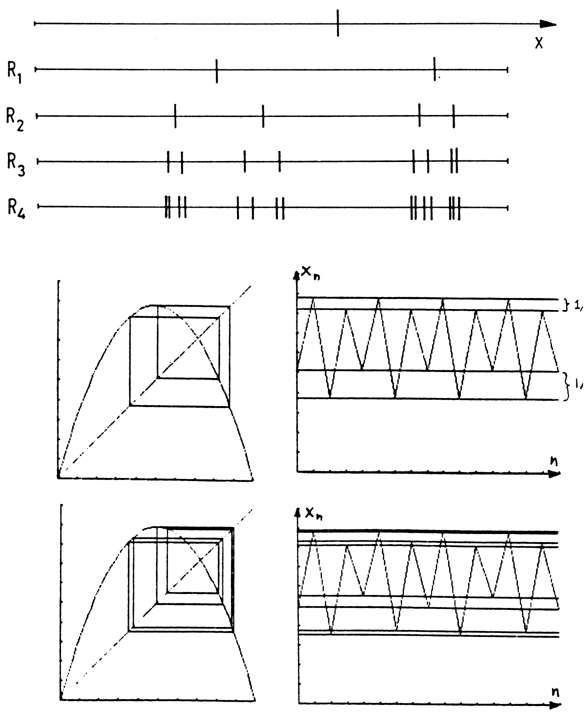

Figure 3.5: (top) (Schuster, Fig. 33, pg. 56) Adjacent pairs of points on the 2^n orbit in the period doubling cascade show a scaling relation. (bottom) (Cvitanovic, Fig. 6.1, pg. 22) The Feigenbaum attractor is a two scale fractal. The scales are $\lambda_1 = 1/\alpha$ and $\lambda_2 = 1/\alpha^2$.

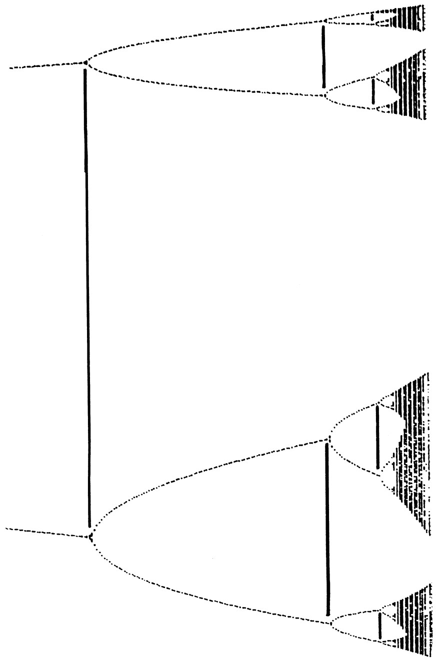

Figure 3.6: The beginning of the bifurcation diagram is shown. At the bifurcation $2^n \rightarrow 2^{n+1}$, 2^{n-1} segments are drawn which connect adjacent points on the 2^n orbit.

3.3.4 Two Scale Fractal Dimensions

Fractal dimension is not generally a constant. To illustrate this idea, we consider a unit square which is mapped into itself according to the following rules. In the generating step, two images are created. In the first image, the x axis $y = 0$ is mapped to itself, the upper side at $y = 1$ is mapped to the parabola $y = 0.01 + 2 \times (x - 0.7)^2$. The x-value is unchanged, and y values are linearly scaled between the boundaries. In the second image, the side $y = 1$ is mapped to itself and the side $y = 0$ is mapped to the straight line $y = 0.99 - 0.80 * x$. The boundaries of these scaling regions are shown with light lines in Fig. 3.8. The fractal dimension (in the y direction) varies as a function of x. The dimension is shown by the heavy line in this figure. A histogram of the fractal dimension distribution for this fractal is shown in Fig. 3.9.

The fractal dimension of the two scale fractal built up by the generating step shown in Fig. 3 .8 is

$$\text{Dimension} = 1 + \langle d \rangle \tag{3.2}$$

The 1 comes from the x-direction, which is smooth. The fractal structure is

```
c       program fracdim.f January 30, 2001
c       This program computes the fractal dimension of the
c       Feigenbaum attractor by the 'divide and conquer' method.
      implicit none
      integer i
      real*8 lam1,lam2
      real*8 alpha,delta,x,y
c       begin
      alpha = 2.50290 78750 95892 8485 ! input data
      delta = 4.66920 16091 029
      lam1 = 1.0/alpha ! establish scaling
      lam2 = 1.0/alpha**2
      dmin = 0.0 ! initialize
      dmax = 1.0
      do i=1,25 ! begin divide and conquer
      d = 0.5*(dmin+dmax)
      y=lam1**d + lam2**d - 1
      if(y.gt.0.0)dmin=d
      if(y.lt.0.0)dmax=d
      end do !! end divide and conquer
      write(*,'(2x,2f12.8)')d,y ! output result, error
      stop
      end
```

Figure 3.7: This short FORTRAN code computes the fractal dimension of the Feigenbaum attractor to 25 bits: $d = 0.52450\ 83040...$.

only in the y direction. Since the fractal dimension varies along the x-axis, the average dimension $\langle d \rangle$ is taken. The average is computed by interpreting the histogram in Fig. 3.9 as a probability distribution: $\langle d \rangle = \int_0^1 z\rho(z)dz$.

3.4 Other Dimensions

A number of other dimensions have been introduced in an attempt to distinguish geometry from dynamics. Almost all of these are based on the invariant measure over a strange attractor. Recall that this is defined as

$$\mu_i = \mu(x_0, C_i) = \lim_{T\to\infty} \frac{\eta(x_0, C_i, T)}{T} \tag{3.3}$$

Here x_0 is an initial condition for the dynamics, C_i is box i in a very refined partition of the phase space, T measures the temporal length of a trajectory, and η measures the total time the trajectory is in cube C_i.

Remark: The quantities μ_i, or their limits $\rho(x)$, are called *measures.* They are invariant measures if $\rho(f(x)) = \rho(x)$ for all x. Invariant measures are closely

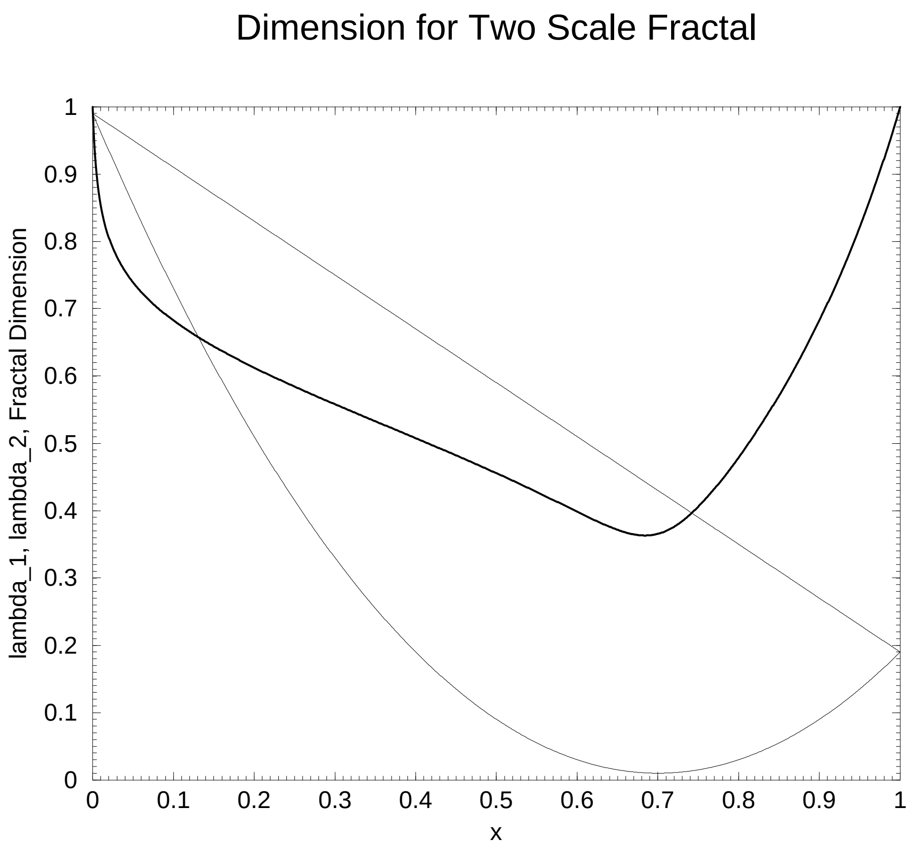

Figure 3.8: The fractal dimension of a two-scale fractal is plotted as a function of position along the x-axis. The two scales are x-dependent. They are the distance below the parabola and the distance above the straight line. The third curve is the fractal dimension.

related to discussions of ergodicity: the equality of time averages with space averages for almost all initial conditions. The *ergodic hypothesis* is usually assumed as a foundation for statistical physics. The existence of invariant measures is a necessary but not sufficient condition for the proof of the *ergodic theorem*. Other conditions ("irreducibility" in some sense) are necessary and not usually met in statistical physics.

3.4.1 Information Dimension

The information dimension is defined as

$$H = \lim_{\epsilon \to 0} \sum_i -\mu_i \log \mu_i \tag{3.4}$$

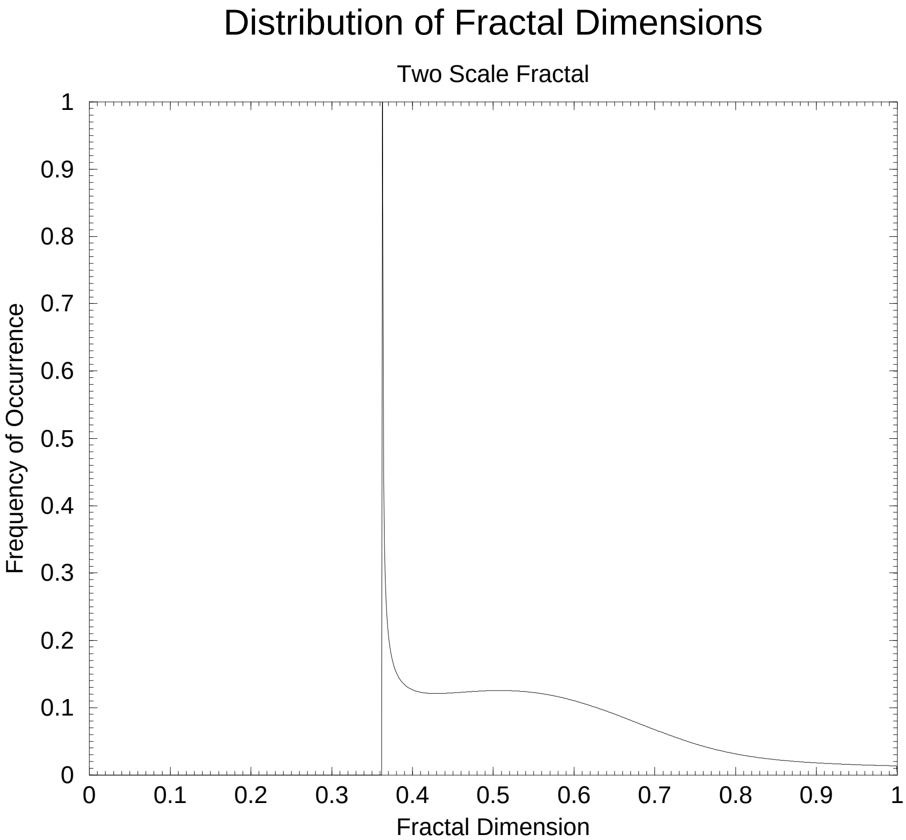

Figure 3.9: Histogram of the relative occurrence of fractal dimension of the two scale fractal is plotted as a function of the fractal dimension. The van Hove singularity is characteristic of a quadratic turn-around, and occurs in one dimensional Quantum Mechanical lattice models.

This is the Shannon (Boltzmann) entropy function.

3.4.2 Correlation Dimension

The correlation dimension is the fractal dimension which is most often used in the analysis of data. It is defined as follows. Count the number of points within a distance ϵ of each other:

$$N(\epsilon) = \sum_{i \neq j} \sum_{j} \Theta(\epsilon - |\mathbf{x}_i - \mathbf{x}_j|) \tag{3.5}$$

In this expression, $\mathbf{x}_i$ are points on an attractor in an n-dimensional phase space, $\Theta(y)$ is the Heaviside theta function: $\Theta(y) = 0$ if $y < 0$ and $\Theta(y) = 1$ if $y \geq 0$ (it is the integral of the Dirac delta function: $\Theta(y) = \int_{-\infty}^{y} \delta(x)dx$). This number

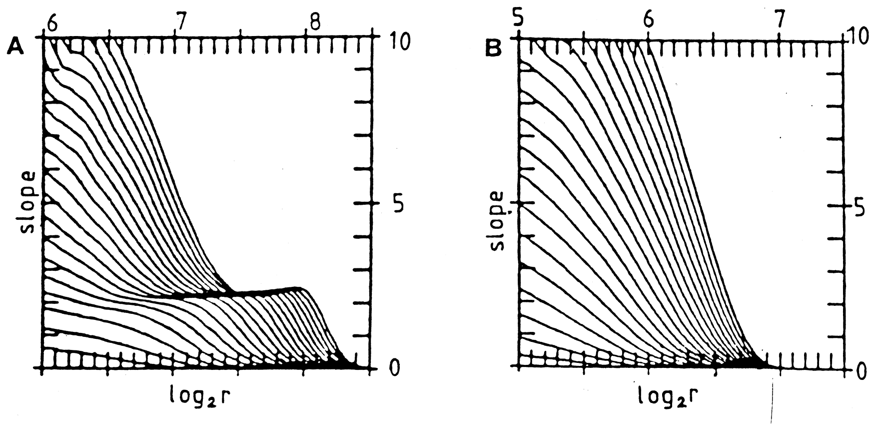

Figure 3.10: (BMC '93, pg. 36) Correlation dimension computations are shown for the X-ray binary Her X-1/HZ Her (A) and background data (B). The embedding dimension ranges from one (bottom curve, both cases) to 20 (top). The correlation integral is capable of distinguishing deterministic chaos from noise.

decreases as ϵ decreases. One hopes that this number decreases exponentially, $N(\epsilon) \sim \epsilon^{d_c}$. If so, then the ratio

$$d_c = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)} \tag{3.6}$$

should (might) exist. This limit defines the correlation dimension. The correlation dimension is generally not the same as either the box counting dimension or the information dimension.

Some samples of the use of this statistic are shown in Figs. 3.10 and 3.11. Fig. 3.10(a) is a correlation dimension calculation for data from the X-ray binary Her X-1/HZ Her. The scalar time series data are embedded as vectors in spaces of dimension 1, 2, $\cdots$, 20 (one curve for each dimension, from bottom to top). The correlation integral is carried out, and its slope is plotted as a function of ϵ. Some of the curves converge to a more or less constant slope over a limited region of the size parameter ϵ. The converged slope is interpreted as the correlation dimension. In Fig. 3.10(b) the same computation is repeated on background data, assumed to be gaussian noise. This series of 20 curves behaves remarkably different. If the correlation dimension computation does not provide a convincing quantitative value for a dimension, at least it provides a mechanism to distinguish between processes with low dimensional deterministic structure and those without.

Fig. 3.11 shows two additional attempts to determine correlation dimensions. In these cases, embeddings of dimensions 1, 2, $\cdots$, 40 were made. The curves on the left were computed for numerically generated data from the Lorenz attractor. The curves on the right were made from data taken on a far infrared

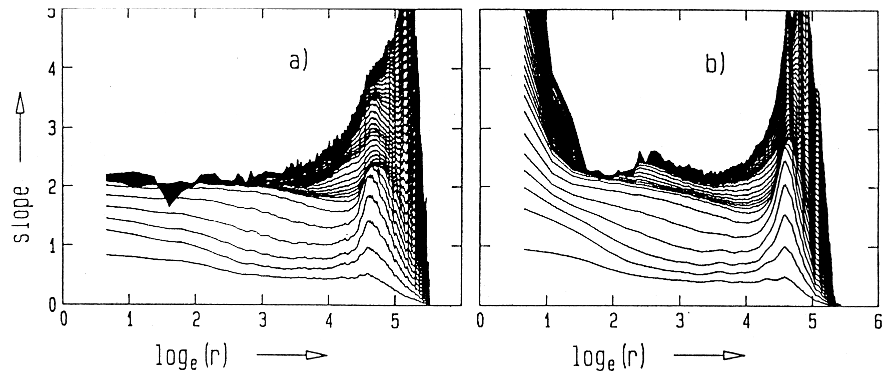

Figure 3.11: (BMC '93, pg. 135) Correlation dimension computations are shown for numerically generated data (Lorenz model) and experimental data (fir laser). Correlation dimensions are computed using embeddings ranging from dimensions one (bottom line) to 40 (top). The peak on the right is not a signal, since the dimension is defined in the limit of ϵ (r) small. A reasonable small r limit is obtained for the numerical data, but not for the experimental data, where the noise daemon kills the computation.

laser. In both cases the behavior at large ϵ is not useful. The definition involves the $\epsilon \to 0$ limit, so the behavior on the right is of little interest, anyway. Computations based on numerically generated data seem to converge to a value slightly above 2 in the small ϵ limit. This is not the case for computations based on experimental data. The problem here is that noise kills the computation. With results like this, it is not too surprising that theorists demand enormously long data sets with unbelieveable signal to noise ratios for correlation dimension computations, while experimentalists view these computations with a jaundiced eye.

3.4.3 D_q Dimensions

There is an entire one-parameter family of dimensions based on the invariant measure μ_i. These dimensions are defined by the limit

$$D_q = \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\log(\sum_i \mu_i^q)}{\log(1/\epsilon)} \tag{3.7}$$

Here ϵ is the diameter of the largest box C_i in the partition of the phase space.

Three special cases of this dimension have already been introduced:

D_0	Box Counting Dimension
D_1	Information Dimension
D_2	Correlation Dimension

The information dimension can be obtained from the definition above by taking a delicate limit at $q \to 1$.

This family is monotonic decreasing, or at least monotonic non-increasing. For example, $D_0 \geq D_1 \geq D_2$. A plot of D_q vs. q is shown in Fig. 3.12 for the Feigenbaum attractor. The scaling function $f(\alpha)$ is shown for this spectrum in Fig. 3.13.

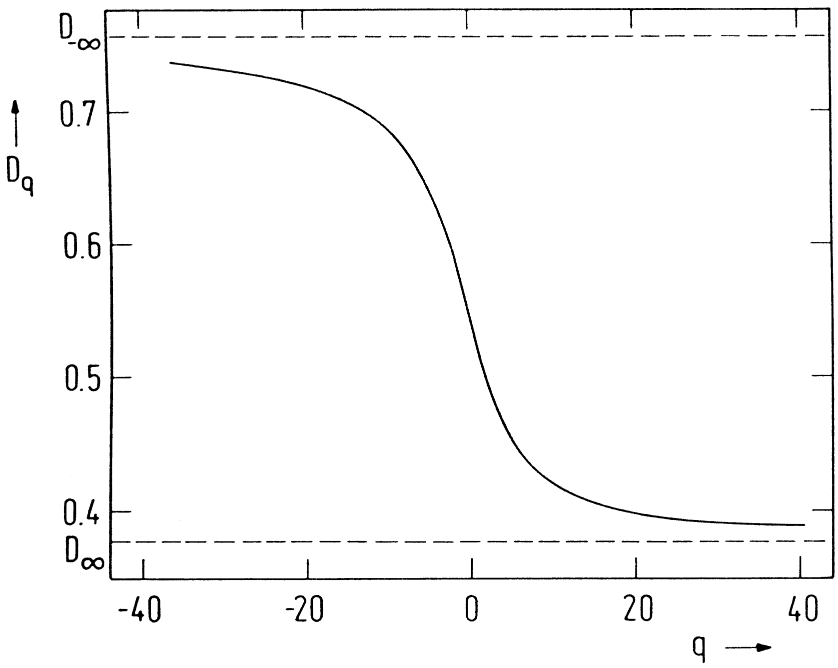

Figure 3.12: (Schuster, Fig. 84, pg. 130) The one parameter family of dimensions D_q is plotted vs. q for the Feigenbaum attractor.

3.4.4 Multifractal Scaling and $f(\alpha)$

A fractal for which the spectrum D_q of dimensions is not constant, but depends on q, is called a multifractal. A formalism has been developed to describe multifractals. In this formalism, the multifractal scaling function $f(\alpha)$ plays a prominent role. We introduce this function as follows.

The invariant measure μ_i scales with the smallness parameter ϵ with some power law dependence: $\mu_i \sim \epsilon^{\alpha_i}$. It is possible to build up a histogram of the distribution over the exponents α in the usual way. Call this histogram $N(\alpha)$.

Then the multifractal scaling function $f(\alpha)$ is related to the histogram $N(\alpha)$ as follows:

$$f(\alpha) \sim -\log N(\alpha) \tag{3.8}$$

A theory can be built up more rigorously as follows. Define a function τ from D_q as follows:

$$\tau = (q-1)D_q = -\lim_{\epsilon \to 0} \frac{\log(\sum_i \mu_i^q)}{\log(1/\epsilon)} \tag{3.9}$$

Then define α as the slope of τ:

$$\alpha = \frac{d}{dq}\left[(q-1)D_q\right] = \frac{d\tau}{dq} \tag{3.10}$$

Finally define $f(\alpha)$ as the Legendre transform of τ:

$$f(\alpha) = q\frac{d\tau}{dq} - \tau(q) \tag{3.11}$$

The monotonic decreasing property of D_q translates into a concavity property on $f(\alpha)$. In Fig. 3.12 we show the monotonic decreasing spectrum of fractal dimensions for the Feigenbaum attractor. In Fig. 3.13 we show the multifractal scaling function $f(\alpha)$ for the same attractor. Because of the close relation of the two (the transformations are invertible), many properties of one are reflected in the properties of the other, as suggested in Fig. 3.13.

The multifractal scaling function is difficult to determine from experimental data, and in the end provides little leverage for distinguishing one dynamical system from another.

3.4.5 Thermodynamic Formalism

There is a 1-1 relationship between the multifractal scaling formalism and classical thermodynamics which is as breathtaking in its elegance and beauty as it is useless in distinguishing among different mechanisms for generating fractal strange attractors, let alone distinguishing among different dynamical systems.

The identifications are shown in the Table below:

Thermodynamics	Multi – Fractal Formalism
β	q
$-\log Z = \beta F$	$\tau = (q-1)D_q$
$U = \frac{\partial}{\partial\beta}(-\log Z)$	$\alpha = \frac{d}{dq}(\tau)$
$S = \beta U + \log Z$	$f(\alpha) = q\alpha - \tau$
$\frac{\partial S}{\partial U} = \beta$	$\frac{df}{d\alpha} = q$

Here we use standard nomenclature for the thermodynamic functions: $\beta = 1/kT$, Z is the partition function, U is the internal energy, F is the Gibbs free energy, and S is the entropy.

3.4.6 Lyapunov Dimension

The most useful of all the dimensions is the Lyapunov dimension. It is not based on geometery, but rather on dynamics. It is defined in terms of Lyapunov exponents, which describe the stability of the dynamical system. Since all of the fractals discussed in this chapter are geometric and have no dynamical origins, it is not possible to define a Lyapunov dimension for any of them.

Unfortunately, we have not yet reached the point where we are able to define the Lyapunov exponent of a dynamical system. The definition must wait.

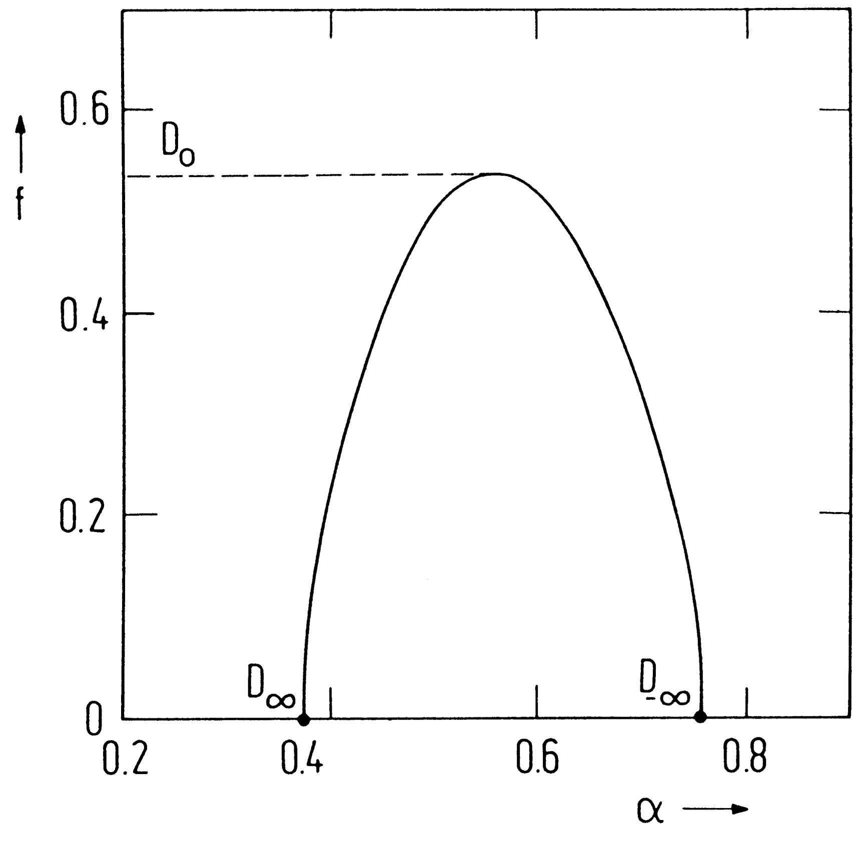

Figure 3.13: (Schuster, Fig. 84, pg. 130) The scaling function $f(\alpha)$ is plotted vs. α for the Feigenbaum attractor.