CATASTROPHE THEORY

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INTRODUCTION

Push on something. It will move. Push just a little bit harder and it will move just a little bit more. This incremental response to incremental stresses is very typical. But under rare conditions a small increase in the level of stress will produce a large and dramatic response. Such a response is called a "catastrophe." This kind of behavior has been summarized succinctly in the phrase "the straw that broke the camel's back."

Although this phenomenon occurs under rare conditions, it is also "typical." That is, although it is unlikely that any *particular* straw will break the camel's back, it is certain that after enough straw has been loaded, *some* straw *will* break the camel's back.

Situations in which gradually increasing stress leads to gradually increasing response, followed by a sudden catastrophic jump to a qualitatively different response state, are all too common. Many examples can be given:

1. Under gradually increased loading a bridge sags to a greater degree, followed by a sudden collapse under the last bit of stress (Zeeman, 1977; Poston and Stewart, 1978; Gilmore, 1981).
2. As temperature is gradually decreased at

Encyclopedia of Applied Physics, Vol. 3

1-56081-062-9/92/$5.00+.50

constant pressure, a gas gradually contracts until a certain temperature is reached, at which it condenses to its liquid (or solid) state, with a sudden and very large change in its volume (Zeeman, 1977; Poston and Stewart, 1978; Gilmore, 1981).

3. Slow variation of the aileron trim settings (wing flaps) of an airplane leads to slow change in the attitude of the plane, until a certain threshold value is passed, at which point there occurs a large change in aircraft attitude (Gilmore, 1981).
4. When a glass tube filled with a helium–neon or carbon dioxide–neon mixture is set inside a cavity formed by two highly reflecting mirrors and a gradually increasing current is passed through it, the amount of incoherent light emitted gradually increases. After a current threshold is passed, the light which is emitted increases rapidly in intensity and coherence: the laser has turned "on" (Poston and Stewart, 1978; Gilmore, 1981).
5. The amount of sunlight falling on the Earth varies gradually over about 1000 years due to variation of the Earth's orbital parameters. Sudden dramatic changes leading to the occurrence and disappearance of the Ice Ages occur during this period and seem to be precipitated only by this gradual variation of the Earth's orbital parameters (Gilmore, 1981).
6. It is even possible to conceive of small causes producing big effects in biological, economic, social, and political systems (Thom, 1975; Zeeman, 1977; Poston and Stewart, 1978).

Catastrophes are widespread, occurring throughout all fields in the scientific and engineering disciplines, and even beyond. Moreover, the mathematical description of a catastrophe follows the same procedure and draws from the same restricted set of functions, independent of whether the catastrophe occurs in the area of physics, chemistry, structural engineering, aircraft dynamics, climate dynamics, etc. The mathematical description of catastrophes, involving mathematical functions called elementary catastrophes, provides a language in terms of which discontinuous phenomena can adequately be described.

It is easy to imagine how to describe systems in which a small push gives rise to a small response. One expects the mathematics of continuity, the calculus of Newton, to be applicable. One also expects that linearization about the local state (linear response function) will give a quantitative estimate of how much the system will respond to a small push.

But what of the mathematics of discontinuity? How does one describe catastrophes? Is it necessary to give up the ideas of continuity?

Roughly speaking, the state of a system is an equilibrium—in fact, a stable equilibrium. By changing some external parameters called control parameters (stress, loading), the equilibrium is displaced. A small change in these parameters usually results in a small displacement of the equilibrium. Sometimes small parameter changes result in the appearance of new equilibria or the disappearance of old equilibria. It is the latter instance in particular that can lead to a catastrophic sudden jump. A systematic study of catastrophes is closely related to a systematic study of equilibria, and especially the appearance and disappearance of equilibria.

Families of functions depending on (control) parameters are called catastrophes when the number of equilibria they possess changes as the parameters are varied. There are only a small number of catastrophes. A very small number have been used to model sudden jumps in physical systems. We will study the mathematics of catastrophes in two steps:

1. We first study functions representing situations in which the number of equilibria is about to change. This occurs when two or more equilibria occur at the same point (become degenerate).
2. We then study the effects of perturbations on the degenerate equilibria. In fact, we identify the simplest perturbation which can reproduce the effects of the most general perturbation.

That both these programs can be carried out successfully is remarkable.

A major obstacle in applying the mathematics of catastrophe theory to physical systems is in identifying the underlying catastrophe. A multiplicity of phenomena occur in the presence of a catastrophe (catastrophe flags). The occurrence of any one of these is an indication that others are present and can be found, and that a catastrophe is ultimately responsible for all. The catastrophe flags are easy to recognize and provide an inordinate amount of information about the underlying catastrophe. This information includes the type of catastrophe, a rough indication of where the sudden jump may occur, and—most important—how to avoid it if that is a desirable objective.

This article is organized into three parts. Section 1 describes what catastrophe theory is, Sec. 2 describes how the catastrophe functions are constructed, and Sec. 3 describes how catastrophe theory is applied to the description of phenomena that occur in the science and engineering disciplines.

In Sec. 1 we describe a progression of three theorems of elementary calculus. These theorems describe local standard forms for functions in the neighborhood of a point. The first, the implicit function theorem, tells us that a function can be replaced by its linear approximation when its slope is nonzero at a point. The second theorem, the Morse lemma, tells us that under suitable conditions a function can be well approximated by a quadratic form in the neighborhood of an equilibrium. The third result, the Thom splitting lemma, describes what happens when the "suitable conditions" required above are not satisfied. In this case two or more equilibria occur very close together (are degenerate), making it possible for a small perturbation either to split or annihilate the equilibria. This change in the number of equilibria is a "catastrophe." Tables containing Thom's list of elementary catastrophes and a complete list of all elementary catastrophes are provided. This is followed by a discussion of the geometric properties of the very simplest catastrophe functions. This section provides a clear answer to one of the questions posed above: it is possible to describe discontinuous phenomena without giving up the ideas of continuity. In fact, this mathematics of discontinuity is an essential part of the calculus of Newton.

In Sec. 2 we compute some explicit catastrophe functions. The first example starts with a family of functions depending on one state variable and two control parameters. Following an algorithmic procedure, we reduce this to a standard form in the neighborhood of its most degenerate equilibrium. This degenerate equilibrium is then perturbed in order to determine how these critical points can be created and annihilated as a function of the control parameters. The algorithm developed to effect this reduction to normal form is then summarized and applied to a more complicated family of functions.

In Sec. 3 we address the question of how to apply the mathematics of catastrophe theory to real world processes that exhibit discontinuous phenomena. Two widely adopted conventions are first described. These are assumptions about the mathematical characterization of the equilibrium state of a physical system: whether it is determined by a local minimum or the global minimum of a potential, and the conditions under which a jump from one minimum to another occurs. The mathematics tells us only where, how many, and what type of equilibria a catastrophe function possesses; the convention isolates the physically important equilibria. When a physical system exhibits a catastrophe, a multiplicity of phenomena occur. It is useful to be able to recognize them—particularly if one wants to avoid the physical catastrophe (e.g., bridge collapse). These phenomena are called catastrophe flags. The presence of any one is an indication that the others are present. Their recognition provides a great deal of information about the underlying catastrophe. The use of catastrophe conventions and catastrophe flags is illustrated in the context of an important example in Sec. 3.3. This illustrates unexpected dangers which may arise in the design of structures following standard optimization criteria employed to reduce costs. In Sec. 3.4 we indicate how the elementary catastrophes may make their appearance within the broader program of catastrophe theory, in the field of dynamical systems theory. That is, the fold catastrophe appears in the guise of saddle-node bifurcations while the cusp catastrophe appears in the guise of pitchfork and Hopf bifurcations.

In a short Appendix we outline the early turbulent and confusing history of catastrophe theory, which was at one point heralded as the greatest advance in mathematics since the development of the calculus by Newton.

1. WHAT IT IS

In Sec. 1 of this article we describe the enormous mathematical program called catastrophe theory and the much smaller and more manageable mathematical program called elementary catastrophe theory (Sec.

1.1). This latter program is not only the starting point for the study of the larger program, but also a continuation of a very important and rather simple program in elementary calculus. This is the program of determining and classifying the standard, or canonical, forms that functions can assume in any of their neighborhoods.

The first two stages in this program are well known at an intuitive level (Sec. 1.2). These are the implicit function theorem and the Morse lemma, which are the mathematical justifications underlying the approximation of a function by a linear function at a point where its slope is nonzero, or by a quadratic form at an equilibrium (where the slope is zero). The third stage, the Thom splitting lemma, describes what happens when two or more equilibria become degenerate. Under this condition a perturbation can either split the equilibria or annihilate them. Change in the number of equilibria is closely related to the occurrence of sudden jumps in physical systems.

Functions that describe sudden jumps, or changes in the number of equilibria of a system, are called catastrophes. There is a small number of elementary catastrophes. They are classified in Sec. 1.3 and presented in Tables 1 and 2 (Sec. 1.4). Since the number of catastrophes is small, the properties of each can be studied in detail (Sec. 1.5); the results are then directly applicable to any physical system described by that mathematical function. The geometric properties of the two simplest of the elementary catastrophes are studied in detail in Sec. 1.6. In Sec. 1.7 we make quantitative the statement that a small push will displace an equilibrium by just a little bit by computing the linear response function for a large class of physical systems, and showing that this function diverges precisely when a sudden jump is imminent.

1.1 The Program of Catastrophe Theory

Catastrophe theory is a program. The purpose of the program is to determine how the qualitative properties of solutions of equations change as the parameters that appear in the equations change (Gilmore, 1981).

It often happens that small changes in the values of parameters that appear in equations produce only small quantitative changes in the solutions of the equations. However, there may be parameter values for which a small change, either in parameter values, initial conditions, or boundary conditions, produces a large quantitative change in the solutions to these equations. Large quantitative changes in solutions describe qualitative changes in the behavior of the system. Catastrophe theory is concerned with determining the parameter values at which qualitative changes occur in solutions of equations described by parameters (Thom, 1975; Zeeman, 1977; Poston and Stewart, 1978; Gilmore, 1981).

This is an ambitious and difficult program. For example, for systems of equations of the form

$$F_\alpha(x,x',t;c)=0, \tag{1}$$

where F_α is a set of functions; x is an n-vector, $x=(x_1, x_2,\ldots,x_n)\in R^n$, called a state vector; c is a k-vector, $c=(c_1, c_2,\ldots,c_k)\in R^k$, called control parameters; and $'=d/dt$, there are no general results. When the set of equations is restricted to the simpler form of coupled nonlinear first-order ordinary differential equations (also called dynamical systems) of the form

$$x_i'=f_i(x,t;c), \tag{2}$$

very little can be said in general. Many results are known when $n=2$ and f is independent of t. A few results are known when $n=2$ and the forcing term is periodic, $f(x,t;c)=f(x,t+T;c)$. Much less is known when $n=2$ and f is not periodic. The case $n>2$ invites a lifetime of work.

When the forcing function in the dynamical system equations (2) is independent of time and can be written as the gradient of some potential,

$$f_i=-\partial V(x;c)/\partial x_i, \tag{3}$$

then the system

$$x_i'=-\partial V(x;c)/\partial x_i \tag{4}$$

is called a gradient dynamical system. For such systems many results are available.

The qualitative properties of a gradient dynamical system can be constructed by investigating the phase-space portrait of its flow. This can be done by plotting the value of the potential as a function of the phase-space coordinates x_i. The phase-space flow is "downhill" on the potential function. It is easily determined in the neighborhood of each equilibrium, or critical point, independent of the stability of the equilibrium. The local flow portraits around each critical point can then be pasted together to determine a global phase portrait. The potential

$$V(x,y;a,b)=\tfrac{1}{4}x^4+\tfrac{1}{2}ax^2+bx+\tfrac{1}{2}y^2 \qquad (5)$$

is illustrated in Fig. 1, together with the phase-space portrait in the state-variable space.

Small changes in the control-parameter values (a,b) typically produce small changes in the location of the critical points. In turn, this produces only small quantitative changes, and therefore no qualitative change, in the phase-space portrait. Qualitative changes will only occur when changes in the control-parameter values result in changes in the number of critical points. This number can change only when two or more critical points coalesce and annihilate or, viewed from the other direction, two or more critical points are created in phase space and then move apart from each other as the control parameters are varied.

Elementary catastrophe theory is the study of how the critical points of a potential, $V(x;c)$, move about, coalesce and annihilate each other, or are created and disperse from each other, in state space $x\in R^n$ as the control parameters $c\in R^k$ are varied.

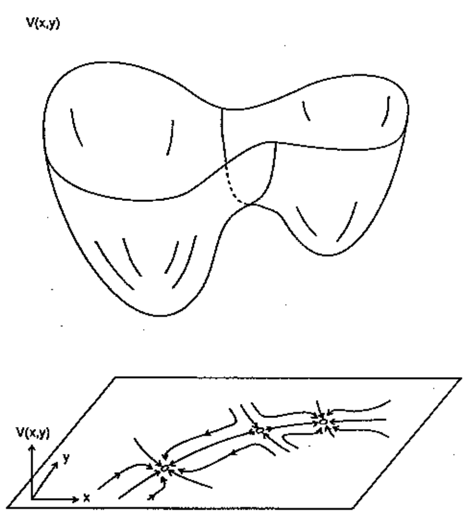

FIG. 1. The dynamics of a gradient system are governed by a potential. The potential of Eq. (5) is shown, together with the flows to and from the stable and unstable equilibria. These are projected down into the *x-y* plane.

1.2 Three Theorems from Elementary Calculus

Elementary catastrophe theory is found at the intersection of two lines of mathematical development, one old, one new. On the one hand, it is the latest development in the quest, in elementary calculus, for standard local forms for functions. On the other hand, it is the first result in the quest, in catastrophe theory, for canonical representations of functions that show qualitative changes when control parameters are varied.

Elementary catastrophe theory is the third in a series of reduction-to-standard-form theorems in elementary calculus. The three developments are the implicit function theorem, which depends on the first derivatives of a function; the Morse lemma, which depends on the second derivatives of a function; and the Thom splitting lemma, which depends on the third (and higher) derivatives of a function. Each of these results provides a standard, or canonical, form for a function in the neighborhood of a point.

We summarize these results now.

1.2.1 Implicit Function Theorem. The implicit function theorem tells us that if the slope of a function is nonzero at a point, the function can be represented locally by a linear approximation to that function. In a rough sense, it tells us that it is justified to linearize a function about a point at which its derivative is nonvanishing.

Implicit Function Theorem: Let $f(x)=f(x_1, x_2,\ldots,x_n)$ be a function with nonzero gradient at x_0:

$$\nabla f|_{x_0}\neq 0. \qquad (6)$$

Then it is possible to find a new coordinate system, $y=(y_1,y_2,\ldots,y_n)$, $y=y(x)$, so that

$$f=y_1. \qquad (7)$$

That is, f is equal, after a smooth change of coordinates, to y_1.

1.2.2 Morse Lemma. The Morse lemma takes over where the implicit function theorem leaves off. Suppose the gradient of a function does vanish at a point—what then? Such a point is called an equilibrium, or critical point. Provided that the function has enough "curvature" at the critical point, it can be represented locally by a quadratic form. In a rough sense, the Morse lemma tells us that it is justified to represent a function at an equilibrium by a quadratic form, provided none of the eigenvalues vanish.

Morse Lemma: Let $f(x)=f(x_1,x_2,\ldots,x_n)$ be a function with equilibrium at x_0 and nonsingular stability matrix at x_0:

$$\begin{array}{ll} \nabla f|_{x_0}=0, & \text{equilibrium}, \\ \det[\partial^2 f/\partial x_i \partial x_j]|_{x_0}\neq 0, & \text{nonsingular}. \end{array} \qquad (8)$$

Then there is a smooth change of coordinates, $x'=x'(x)$, so that

$$f=\sum \lambda_i (x_i')^2, \qquad (9)$$

where λ_i are the eigenvalues (all nonzero) of the stability matrix.

A critical point satisfying the conditions (8) is called variously a Morse critical point or an isolated critical point.

The quadratic form (9) can be put into a canonical form by rescaling the coordinates:

$$y_i=|\lambda_i|^{1/2}x_i'. \qquad (10)$$

Under this scale transformation the function at equilibrium assumes Morse canonical form:

$$\begin{array}{l} f=M_i^n, \\ M_i^n=-y_1^2-\cdots-y_i^2+y_{i+1}^2+\cdots+y_n^2. \end{array} \qquad (11)$$

The quadratic forms (11) are called Morse saddles. The Morse saddle M_0^n has a minimum at $y=0$, while M_n^n has a maximum at $y=0$. The remaining Morse saddles M_i^n, $i\neq 0,n$, have equilibria at $y=0$ that are neither maxima nor minima.

1.2.3 Thom Splitting Lemma. The Thom splitting lemma takes over where the Morse lemma leaves off. Suppose the stability matrix of a function is singular at an equilibrium. Then one or more eigenvalues (λ_i) vanish. What then? The Thom splitting lemma tells us that there is a smooth change of coordinates, $x'=x'(x)$, where $x_1',\ldots,x_l'$ are tangent to the eigenvectors with vanishing eigenvalues at the critical point, and $x_{l+1}',\ldots,x_n'$ are tangent to the eigenvectors with nonvanishing eigenvalues at the critical point, so that the function can be broken down into two parts. One part, associated with the nonzero eigenvalues, is simple and can be put into Morse canonical form. The other part, associated with the vanishing eigenvalues, is interesting and has all its second derivatives equal to zero at the critical point. This non-Morse function is the principal object of study in elementary catastrophe theory.

Thom Splitting Lemma: Let $f(x)=f(x_1,x_2,\ldots,x_n)$ be a function with equilibrium and singular stability matrix at x_0:

$$\begin{array}{ll} \nabla f|_{x_0}=0, & \text{equilibrium}, \\ \det[\partial^2 f/\partial x_i \partial x_j]|_{x_0}=0, & \text{singular}. \end{array} \qquad (12)$$

If the stability matrix has exactly l vanishing eigenvalues, then there is a smooth change of coordinates, $x'=x'(x)$, so that

$$\begin{array}{l} f(x)=f_{\mathrm{NM}}(x_1',\ldots,x_l')+M_i^{n-l}(x_{l+1}',\ldots,x_n'), \\ \partial^2 f_{\mathrm{NM}}/\partial x_i \partial x_j|_{x_0}=0, \quad 1\leqslant i,j\leqslant l. \end{array} \qquad (13)$$

The Thom splitting lemma can be proven by the methods of elementary calculus. It tells us that when the Morse lemma is not applicable, the function can be split into two functions, a "good" function in $n-l$ coordinates which can be put into Morse canonical form and a "bad," or non-Morse, function of l variables which bears further scrutiny. It tells us nothing about the non-Morse function except that its Taylor series expansion about the critical point begins with at least third-degree terms.

We emphasize here that the three results of elementary calculus, the implicit function theorem, the Morse lemma, and the Thom splitting lemma, depending on first, second, and third derivatives, are local in nature. The theorems do not provide an estimate for the size of the neighborhood for which the statement of the result is true.

1.3 Thom Classification Theorem

For a typical function, $f(x)$, the gradient at a random ("typical") point will be nonvanishing, so that the implicit function theorem is applicable. There are, however, typically isolated points at which the gradient vanishes. At such points the stability matrix is typically nonsingular, so that the Morse lemma is applicable. How, then, does it come about that the machinery of elementary catastrophe theory becomes useful?

When the function depends on control parameters c as well as state variables x, $f=f(x;c)$, then the eigenvalues of the stability matrix at a critical point, $x_0=x_0(c)$, depend on the control-parameter values: $\lambda_i=\lambda_i(c)$. As a result, there may be choices of the control-parameter values that annihilate one or more of the eigenvalues.

As a result, the structure of the non-Morse function in Eqs. (13) will depend on control parameters. The Thom classification theorem, which is outside the scope of calculus (elementary or otherwise), provides a further resolution of the non-Morse function into two functions. One of these, the catastrophe germ, depends only on the l state variables $y_1, y_2, \ldots, y_l$ and summarizes the nature of the singularity at the non-Morse critical point. The other function, the universal perturbation, is a function of both the l state variables and k control parameters. This function summarizes what can happen to the singularity, or degenerate critical point, under the most general possible ("universal") perturbation.

Thom Classification Theorem: Let $f_{\mathrm{NM}}(y; c) = f(y_1, \ldots, y_l; c_1, \ldots, c_k)$ be a non-Morse function of l state variables and k control parameters. Then there is a smooth change of coordinates so that

$$f_{\mathrm{NM}}(y;c) = \mathrm{Cat}(l,k). \qquad (14)$$

The elementary catastrophe function, $\mathrm{Cat}(l,k)$, is the sum of two terms:

$$\mathrm{Cat}(l,k) = \mathrm{CG}(l) + \mathrm{Pert}(l,k). \qquad (15)$$

The catastrophe germ, $\mathrm{CG}(l)$, depends only on the l state variables. All its second partial derivatives vanish at the critical point. The universal perturbation depends on the k control parameters as well as the l state variables. The dependence of $\mathrm{Pert}(l,k)$ on the control-parameter values is linear. For "most" choices of control-parameter values (all but a set of measure zero) the function $\mathrm{Cat}(l,k)$ has isolated critical points.

The Thom classification theorem, like the three results of elementary calculus described in the previous section, is local in nature. The theorem does not provide an estimate of the size of the neighborhood for which the statement of the theorem is true.

1.4 Thom's List of Elementary Catastrophes

Thom's original classification theorem provided a list of the elementary catastrophes (Thom, 1975). A slightly expanded version of this list is provided in Table 1 (Arnol'd, 1981, 1986). This list contains the canonical catastrophe functions for $k<6$ and therefore (cf. Sec. 2.3) $l<3$. This list consists of the classification of the function following the beautiful convention introduced by Arnol'd (1981), the original descriptive name, when it exists (Thom, 1975; Zeeman, 1977; Poston and Stewart, 1978), values for k and l, the catastrophe germ, $\mathrm{CG}(l)$, and the universal perturbation, $\mathrm{Pert}(l,k)$. Thom's original list contained only the seven members with $k \leqslant 4$ (dimension of spacetime) for unsupportable historical reasons.

The catastrophe functions listed in Table 1 are elementary in the sense that all coefficients in the catastrophe germ can be assigned canonical values. There are no free parameters; every coefficient in the catastrophe germ can be given canonical numerical values such as $\pm 1, 0$ by a coordinate change. For example, a term of the form $-3x^4$ in the A_{-3} catastrophe could be transformed to the canonical

Table 1. All catastrophes up to control-parameter dimension five are elementary and are listed below by dimension of control-parameter space.

k	l	Classification	Name	CG(l)	Pert(l,k)
1	1	A_2	Fold	x^3	a_1x^3
2	1	$A_{\pm 3}$	Cusp	$\pm x^4$	$a_1x^2+a_2x^2$
3	1	A_4	Swallowtail	x^5	$a_1x^2+a_2x^2+a_3x^3$
3	2	D_{-4}	Elliptic umbilic	x^2y-y^3	$a_1x+a_2y+a_3x^2$
3	2	D_{+4}	Hyperbolic umbilic	x^2y+y^3	$a_1x+a_2y+a_3x^2$
4	1	$A_{\pm 5}$	Butterfly	$\pm x^6$	$a_1x+a_2x^2+a_3x^3+a_4x^4$
4	2	D_5	Parabolic umbilic	$\pm(x^2y+y^4)$	$a_1x+a_2y+a_3x^2+a_4y^2$
5	1	A_6	Wigwam	x^7	$a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5$
5	2	D_{-6}	Second elliptic umbilic	x^2y-y^5	$a_1x+a_2y+a_3x^2+a_4y^2+a_5y^3$
5	2	D_{+6}	Second hyperbolic umbilic	x^2y+y^5	$a_1x+a_2y+a_3x^2+a_4y^2+a_5y^3$
5	2	$E_{\pm 6}$	Symbolic umbilic	$\pm(x^3+y^4)$	$a_1x+a_2y+a_3xy+a_4y^2+a_5xy^2$

form $-x'^4$ (or $-\frac{1}{4}x'^4$) by an appropriate scale change $x'=\lambda x$.

Thom's list of elementary catastrophes is not a complete list. The complete list is provided in Table 2. There are two infinite series of elementary catastrophes and one finite series. One infinite series, the cuspoids $A_{\pm k}$, depends on only one state variable. The other infinite series, the umbilics $D_{\pm k}$, depends on two state variables. The exceptional elementary catastrophes, $E_{\pm 6}$, E_7, E_8, depend on two state variables (Arnol'd, 1981, 1986; Poston and Stewart, 1978; Gilmore, 1981).

There is a remarkable correspondence between the classification theory of elementary catastrophes and the classification theory for Lie algebras all of whose roots (in the root space diagram) have the same length (Arnol'd, 1981, 1986; Gilmore, 1981). The correspondence is as follows. The phase-space portraits of the elementary catastrophes with maximum number of isolated critical points can be summarized by drawing the flow from each equilibrium to any of the others to which a flow is possible. The phase-space portraits so obtained are exactly the Dynkin diagrams which classify all the simple Lie algebras whose roots have equal length; these are A_{n-1}, D_n, and the exceptional simple Lie algebras E_6, E_7, and E_8. This nomenclature for simple Lie algebras has accordingly been adapted to the classification of elementary catastrophes. For Lie algebras the subscript (e.g., 8 for E_8) denotes its rank; for elementary catastrophes the subscript denotes the number of isolated (complex) critical points generated by an arbitrary perturbation of the function. This is the maximum number of real critical points into which the non-Morse critical point splits under a general perturbation.

Tables 1 and 2 differ in a subtle way, indicating that they are responses to somewhat different questions. The question to which Table 1 responds is: "Up to what control-parameter dimension are all catastrophes elementary, and what are they?" The question to which Table 2 responds is: "For each control dimension $k=1,2,\ldots$, what are the elementary catastrophes?"

The difference between the two tables indicates that for control dimension $k\geqslant 6$ there are catastrophes that are elementary and those that are not, while for $k<6$ all catastrophes are elementary. We will explore what happens at $k=6$ to generate nonelementary (modal) catastrophes as well as elementary catastrophes in Sec. 2.3. Briefly, the result is as follows. To annihilate l eigenvalues of the stability matrix requires $k\geqslant l(l+1)/2$ control parameters. A linear transformation can be used in an attempt to provide canonical values for cubic terms in the Taylor series expansion

Table 2. There are three series of elementary catastrophes. The cuspoids A_k depend on one state variable while the umbilics D_k and the exceptional catastrophes E_k, $k=6,7,8$, depend on two state variables. The subscript k indicates the maximum number of real Morse critical points the catastrophe splits into under an arbitrary perturbation. The number of control parameters required in the universal perturbation is $k-1$.

Symbol	Catastrophe germ	Universal perturbation
$A_{\pm k}$[a]	$\pm x^{k+1}$	$\sum_{j=1}^{k-1} a_j x^j$
$D_{\pm k}$	$x^2y \pm y^{k-1}$, k even $\pm(x^2y+y^{k-1})$, k odd	$\sum_{j=1}^{k-3} a_j y^j + \sum_{j=k-2}^{k-1} a_j x^{j-(k-3)}$
$E_{\pm 6}$	$\pm(x^3+y^4)$	$\sum_{j=1}^{2} a_j y^j + \sum_{j=3}^{5} a_j xy^{j-3}$
E_7	x^3+xy^4	$\sum_{j=1}^{4} a_j y^j + \sum_{j=5}^{6} a_j xy^{j-5}$
E_8	x^3+y^5	$\sum_{j=1}^{3} a_j y^j + \sum_{j=4}^{7} a_j xy^{j-4}$

[a] $A_{+k}=A_{-k}$ if k is even.

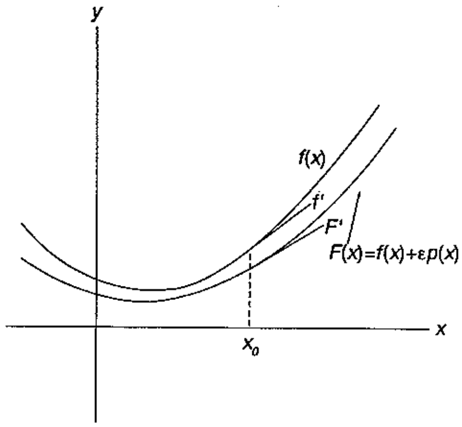

FIG. 2. At a point where the implicit function theorem is applicable, perturbation of a function does not produce a qualitative change in the function.

of the non-Morse function. When the number of cubic coefficients $[(l+3-1)!/(l-1)!3!]$ exceeds the number of degrees of freedom in the $l \times l$ linear transformation (l^2), then it is not possible to assign canonical values to all cubic terms in the Taylor series expansion of the non-Morse function, and the resulting catastrophe germ cannot be elementary. This first occurs for $l=3$ ($\Rightarrow k=6$). For $k \geqslant 6$ there are catastrophes that are elementary and those that are not. For $k<6$ all catastrophes are elementary.

1.5 Why a List of Perturbations is Required

The first two canonical form theorems of elementary calculus are clean and simple. If the function has certain properties at a point, then the canonical form in the neighborhood of the point is provided by the statement of the theorem (implicit function theorem, Morse lemma). By contrast, the third result is not nearly so clean cut. The Thom splitting lemma tells us that we can decompose a function at a non-Morse critical point into the sum of two functions, one Morse, the other interesting. The classification theorem provides a list of the interesting functions by number of state variables (l) and control parameters (k). Why is it that the classification theorem, in addition to providing a list of canonical forms for catastrophe germs, in the spirit of the implicit function theorem and the Morse lemma, also provides a list of canonical perturbations? The reason is that perturbation of the canonical linear form which is provided by the implicit function theorem does not change its qualitative properties. The same is true for the canonical quadratic form which is provided by the Morse lemma. However, perturbation of the canonical singularity CG(l) provided in Tables 1 and 2 produces dramatic changes in its qualitative properties. Different perturbations produce different qualitative changes. The canonical perturbation, Pert(l,k), of each catastrophe germ is the "smallest" function, in the sense of number of control parameters required, which incorporates all distinct qualitative changes produced by all possible perturbations of the catastrophe germ.

We illustrate these statements in Figs. 2–4. In Fig. 2 we show a function, $f(x)$, which satisfies the conditions of the implicit function theorem at x_0. Under a perturbation, $\varepsilon p(x)$, where $p(x)$ is a well-behaved function and ε is a small parameter, the new function, $F(x) = f(x)+\varepsilon p(x)$, also satisfies the conditions of the implicit function theorem at x_0 for ε sufficiently small. Therefore, perturbation of $f(x)$ at x_0 does not change its qualitative properties (perturbation "commutes" with the implicit function theorem).

In Fig. 3 we show a function, $f(x)$, which satisfies the conditions of the Morse lemma at x_0. Under a perturbation, $\varepsilon p(x)$, the new function, $F(x)=f(x)+\varepsilon p(x)$, no longer appears to satisfy the conditions of the Morse lemma at x_0, since typically $p'(x_0)\neq 0$. However, $F'(x_0)=\varepsilon p'(x_0)$ is small for small ε so

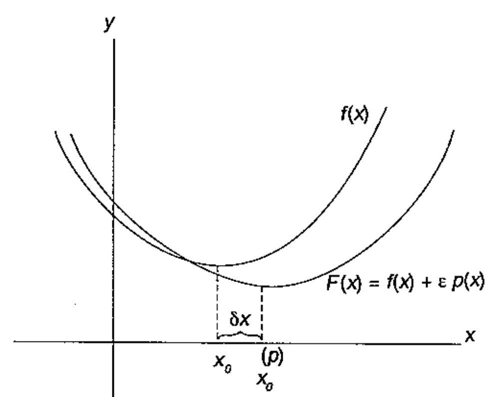

FIG. 3. At a point where the Morse lemma is applicable, perturbation of a function does not produce a qualitative change in the function.

that the implicit function theorem is on the verge of not being applicable. It is more useful to regard the perturbation as moving the location of the critical point to a nearby point, $x_0^{(p)} = x_0 + \delta x_0$. At this perturbed critical point the conditions for the Morse lemma are fulfilled. Thus, a small perturbation at an equilibrium produces only a small displacement of that equilibrium. Not only that, but the curvature or, more generally, the Morse saddle type of the canonical form remains unchanged. As a result, perturbation of a function that satisfies the conditions for the Morse lemma does not produce a qualitative change of the function in the neighborhood of the critical point.

The situation is quite different for a catastrophe germ. In Fig. 4 we plot the catastrophe function A_2: $f(x;a_1) = \frac{1}{3}x^3 + a_1 x$ for three values of the control parameter a_1. For $a_1 = 0$ the catastrophe germ $f(x;a_1=0) = x^3/3$ has a doubly degenerate critical point at $x=0$. The perturbation with $a_1 < 0$ splits this doubly degenerate critical point into two isolated critical points at $x_- = -(-a_1)^{1/2}$ and $x_+ = +(-a_1)^{1/2}$. The perturbation with $a_1 > 0$ removes the critical point altogether. These are the only two qualitatively distinct things that can occur to a doubly degenerate critical point under an arbitrary perturbation. These correspond to scattering of the solutions of $\nabla f(x;a_1) = 0$ from the real axis to the imaginary axis as a_1 increases through zero, with the point of double degeneracy occurring at $a_1 = 0$. It is a remarkable result that the simple perturbation $P(1,1) = a_1 x$ encapsulates all distinct possibilities under generic perturbations.

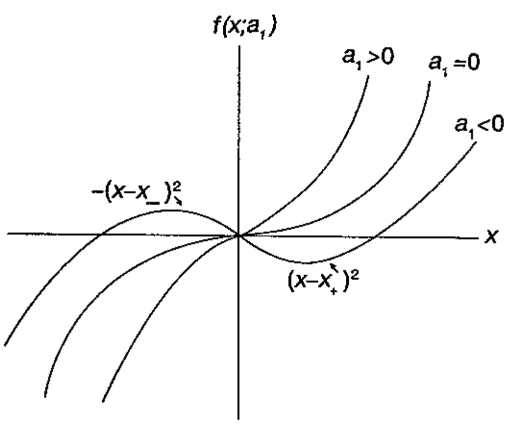

FIG. 4. At a point where a function has a degenerate critical point, so that neither the implicit function theorem nor the Morse lemma is applicable, perturbation of a function produces a qualitative change in the function. In the case shown, perturbation of the function x^3 either annihilates the critical points or splits them into two nondegenerate critical points.

It should now be clear why the catastrophe germs listed in the classification theorem must be accompanied by a list of universal perturbations while the implicit function theorem and the Morse lemma are not encumbered by such baggage. The canonical linear and quadratic forms are invariant under perturbation: perturbation produces no qualitative change. However, the catastrophe germs undergo a wide spectrum of distinct qualitative changes under perturbation. The perturbation functions listed are those of minimal control-parameter dimension which are capable of reproducing the entire spectrum of distinct qualitative changes induced by the most general perturbation.

It is a remarkable result that the control parameters appear linearly in these perturbations.

1.6 Geometry of the Fold and the Cusp

In this section we review the properties of the two simplest elementary catastrophes, the fold catastrophe A_2 and the cusp catastrophe $A_{\pm 3}$. Since the cusp catastrophe A_{+3} occurs more frequently in physical applications than A_{-3} (which is not bounded below), we study specifically the properties of A_{+3}. The properties of A_{-3} are related by appropriate sign changes. We also review a restricted set of properties of the co(ntrol)-dimension three catastrophes A_4, $D_{\pm 4}$.

For the fold and the cusp we study the following properties:

1. typical functions in the family of functions as well as the bifurcation set;
2. location of the critical points;
3. values of the function at the critical points;
4. curvature of the function at the critical points.

We present only the bifurcation set for the three catastrophes A_4, $D_{\pm 4}$.

1.6.1 Geometry of the Fold Catastrophe.

The fold catastrophe is

$$A_2:\ f(x;a) = \tfrac{1}{3}x^3 + ax. \qquad (16)$$

The canonical properties of this function are shown in Fig. 5. In Fig. 5(a) we show members of this family with $a>0$, $a=0$, $a<0$. The bifurcation set is the set of points in the control parameter space at which there is a qualitative change in the nature of the function. This occurs when two or more critical points become degenerate. For the fold catastrophe this consists of the single point $a=0$, at which there is a doubly degenerate critical point at $x=0$.

The location of the critical points, the solution of $\nabla f(x;a)=0$, is shown in Fig. 5(b). The critical points, $x_\pm(a)=\pm(-a)^{1/2}$, have a standard 1/2 power-law dependence on the control parameter a. Note that the critical points $x_\pm(a)$ exist only for $a\leqslant 0$ and that the graph of $x_\pm(a)$ as a function of a is a smooth manifold embedded in the space R^1 (state space) $\times R^1$ (control parameter space). This is a general result. The fold catastrophe derives its name from the shape of its critical set $\nabla f(x;a)=0$, which looks like a curve folded over itself.

The value of the function at the critical points,

$$f_c(x_\pm(a);a)=\pm\tfrac{2}{3}(-a)^{3/2}, \qquad (17)$$

is plotted in Fig. 5(c). This curve has a canonical 3/2 power-law dependence. This graph is not generally a manifold.

The curvature of the function,

$$f''(x_\pm(a);a)=\pm 2(-a)^{1/2}, \qquad (18)$$

is shown in Fig. 5(d). Although the critical curvature happens to be a manifold in the present case, this is not generally true for the remaining catastrophes.

1.6.2 Geometry of the Cusp Catastrophe. The cusp catastrophe is

$$A_{+3}\text{: } f(x;a,b)=\tfrac{1}{4}x^4+\tfrac{1}{2}ax^2+bx. \qquad (19)$$

The canonical properties are shown in Fig. 6. In Fig. 6(a) we show the control-parameter plane $R^2=(a,b)$, various points in this plane, and the function $f(x;a,b)$ evaluated at these points. Within the cusp-shaped region the

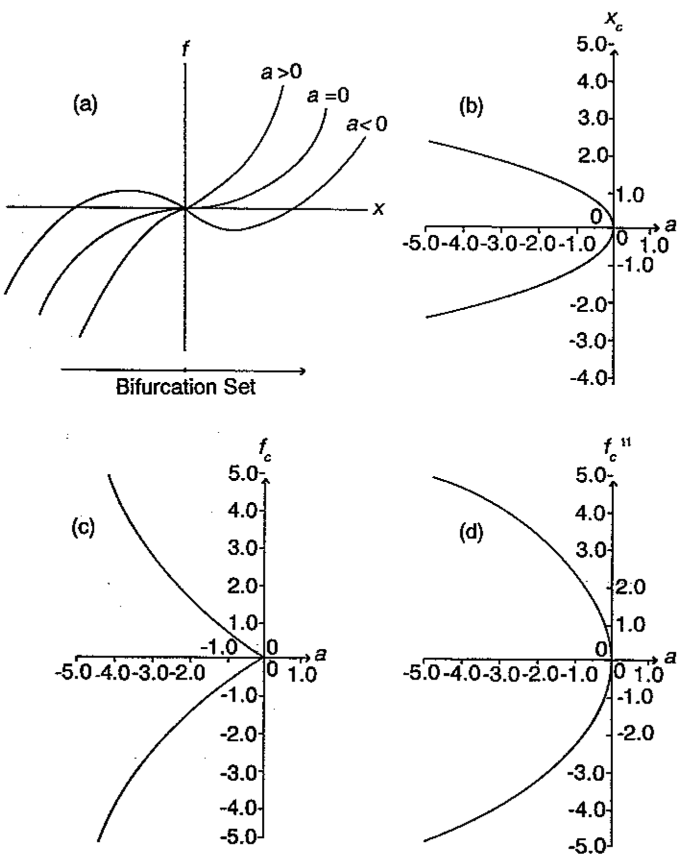

FIG. 5. (a) Members of the fold family of functions $f(x;a)=\frac{1}{3}x^3+ax$ for various values of the control parameter a. (b) Location of the critical points as a function of a. (c) Value of the function at its critical points. (d) Curvature of the function at its critical points.

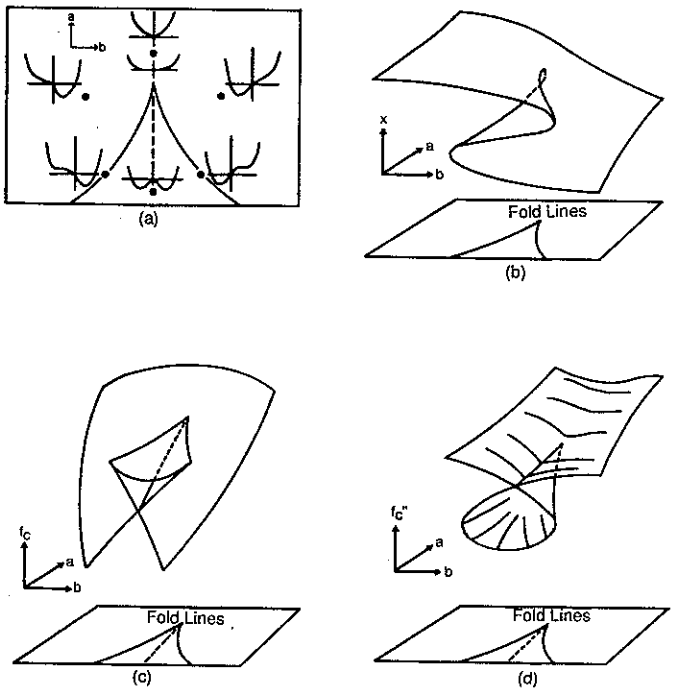

FIG. 6. (a) Members of the cusp family of functions $f(x;a,b)=\frac{1}{4}x^4+\frac{1}{2}ax^2+bx$ for various values of the control parameters a,b. (b) Location of the critical points as a function of position over the control parameter plane. (c) Value of the function at its critical points. (d) Curvature of the function at its critical points.

function has three isolated critical points, two minima separated by a local maximum. Outside the cusp-shaped region the function has only a single minimum. These two regions in the control plane parametrize two qualitatively distinct types of functions. Any path from one region to the other must pass through the cusp-shaped curve, along which there is a doubly degenerate critical point (triply degenerate at the tip of the cusp). This degeneracy occurs when the local maximum collides with one of the two minima. Entering the larger region complementary to the cusp-shaped region, the two degenerate critical points annihilate each other in a catastrophe which is of type A_2.

The cusp-shaped bifurcation set is determined by the condition that a critical point $[\nabla f(x;a,b)=0]$ is degenerate $[\nabla^2 f(x;a,b)=0]$:

$$\begin{aligned}&\text{critical point:}\quad f'(x;a,b)=x^3+ax+b=0,\\ &\text{degenerate:}\quad f''(x;a,b)=3x^2+a=0.\end{aligned}\tag{20}$$

From these two equations we compute the semicubical parabola

$$\begin{aligned}a&=-3x_c^2,\\ b&=2x_c^3,\end{aligned}\tag{21a}$$

along which a critical point is degenerate. The projection of this space curve into the control parameter plane is

$$(a/3)^3+(b/2)^2=0.\tag{21b}$$

This is the bifurcation set shown in Fig. 6(a).

In Fig. 6(b) we show the critical point(s), $x_c(a,b)$, as a function of the control parameters a,b. These points lie on the critical manifold or catastrophe manifold $\nabla f(x;a,b)=0$. Outside the cusp-shaped region there is a single critical point. Over the cusp-shaped region there are three. The middle critical point is the local maximum which separates the two minima. Moving toward the edge of the cusp-shaped region, two of the critical points move together. They collide on the bifurcation set and annihilate each other beyond the bifurcation set. The graph $x_c(a,b)$ in R^1 (state space) $\times R^2$ (control parameter space) is a smooth two-dimensional manifold. The locus of points on this manifold where the tangent is "vertical" is the semicubical parabola (21). From another point of view, the cusp-shaped bifurcation set in R^2 is the projection into R^2 of the fold in the manifold $x_c(a,b)$ in $R^1\times R^2$. The singularity in this catastrophe lies not in the catastrophe manifold itself, which is smooth, but in the projection of this two-dimensional manifold down into the two-dimensional control parameter space. In general, the graph of $\nabla f(x;c)=0$, with $x \in R^n$ and $c \in R^k$, is a smooth k-dimensional manifold embedded in $R^n \times R^k$. The only singularity occurs in the projection of this k-dimensional manifold into the k-dimensional space of control parameters.

In Fig. 6(c) we present the critical function, the value(s) of the function at the critical point(s). This graph is not a manifold because of the sharp corners and self-intersections. The two lower pieces of the graph are the values of the function at the two minima. Where these pieces intersect, the minima are equally deep (Maxwell set, $a<0$, $b=0$, Sec. 3.1). The remaining piece of this graph, which looks like the seat of an Art Moderne chair, is the value of the function at the local maximum. The creases at which the values at the local maximum and minimum join have canonical power-law dependence familiar from the behavior of the Gibbs free energy of a function exhibiting a second-order phase transition (Gilmore, 1981). This canonical power-law dependence is that of the fold catastrophe, namely, 3/2.

The critical curvature, $f''(x_c(a,b);a,b)$, or curvature of the function at its critical point(s), is shown in Fig. 6(d). The curvature is positive at the local minima and negative on the intermediate local maximum. Although there are no creases as in Fig. 6(d), this graph is not a manifold because of the self-intersection. Over the bifurcation set the critical curvature vanishes because the second derivatives vanish and the tangent is "vertical."

1.6.3 Bifurcation Sets for the Three-Dimensional Catastrophes. The geometry of the fold and the cusp was relatively easy to visualize because their graphs could be embedded in low-dimensional spaces: $R^1 \times R^1$ for the fold and $R^1 \times R^2$ for the cusp. Higher-dimensional catastrophes are more difficult to visualize. The catastrophe A_4 should be viewed in $R^1 \times R^3$ while the catastrophes $D_{\pm 4}$ should be viewed in $R^2 \times R^3$. However, the bifurcation sets for these three catastrophes are relatively simple to visualize, since they are embedded in the control-parameter space R^3. These three catastrophes are

$$A_4:\quad f(x;a,b;c)=\tfrac{1}{5}x^5+\tfrac{1}{3}ax^3+\tfrac{1}{2}bx^2+cx,$$

$$D_{+4}:\quad f(x,y;a,b,c)=x^2y+\tfrac{1}{3}y^3+a(y^2-x^2)+bx+cy,$$

$$D_{-4}:\quad f(x,y;a,b,c)=x^2y-\tfrac{1}{3}y^3+a(y^2+x^2)+bx+cy. \tag{22}$$

In each instance the three-dimensional control-parameter space is partitioned into open regions by two-, one-, and zero-dimensional manifolds, components of the bifurcation set on which two, three, and all four of the critical points are degenerate. Within each open region the critical points are isolated; their number and type are unchanged by a sufficiently small perturbation. The number of critical points can change only when passing from one open region to another through the bifurcation set. The bifurcation sets for these three catastrophes are shown in Fig. 7. Shown in each figure is the number of critical points possessed by the catastrophe function in each of the open regions in its control parameter space. The catastrophes A_4, D_{+4} can each have zero, two, or four nondegenerate critical points while D_{-4} can have only two or four nondegenerate critical points.

1.7 Perturbations of Gradient Dynamical Systems

The qualitative properties of a gradient dynamical system are determined by the number, saddle type, and distribution of its critical points. If the critical points are isolated, then the dynamical system is structurally stable against perturbations. If one or more critical points are degenerate, the system is structurally unstable—a perturbation will produce a qualitative change in the properties of the system by splitting or annihilating the degenerate critical points.

As a result it is sufficient to use perturbation theory to describe the effect of a perturbation on a structurally stable system. In the case of a structurally unstable system it is useful to reduce the degenerate critical point to canonical form (a catastrophe) and then discuss the effect of a perturbation by using the catastrophe germ's universal perturbation.

To illustrate the effect of a perturbation in the structurally stable case, we consider a family of potentials, $V(x;c)$, depending on n state variables and k control parameters. As-

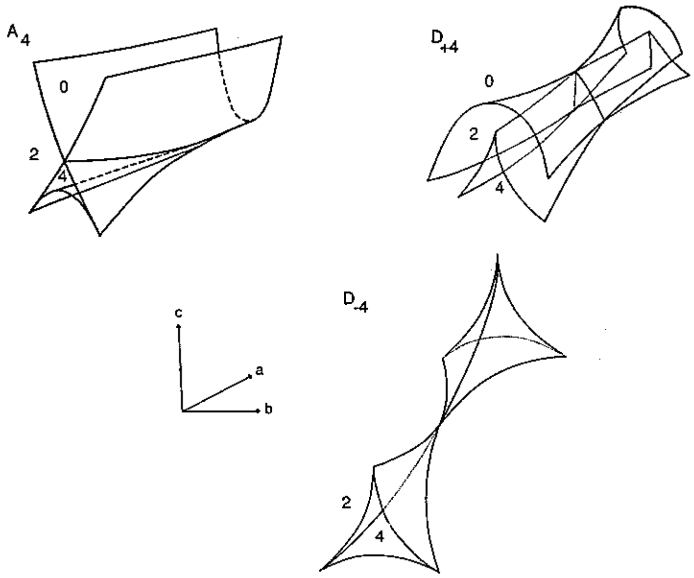

FIG. 7. Bifurcation sets for the three elementary catastrophes [Eq. (22)] of control dimension three as a function of the three control parameters *a,b,c*. The number of isolated critical points is shown in each of the open regions into which the control-parameter space is divided by the bifurcation set.

sume x_0 is a critical point for control-parameter value c_0. What happens under a perturbation $c_0 \to c' = c_0 + \delta c$? We expect that the critical point x_0 will move to a nearby point: $x_0 \to x' = x_0 + \delta x$. The displacement of the critical point as a function of the change in control-parameter value is determined by expanding $V(x_0+\delta x; c_0+\delta c)$ in a Taylor series about x_0, c_0:

$$\begin{aligned} V(x_0+\delta x;c_0+\delta c) \\ &= V(x_0;c_0) + V_i \delta x_i + V_\alpha \delta c_\alpha + \tfrac{1}{2} V_{ij} \delta x_i \delta x_j \\ &\quad + V_{i\alpha} \delta x_i \delta c_\alpha + \tfrac{1}{2} V_{\alpha\beta} \delta c_\alpha \delta c_\beta \\ &\quad + \text{higher-order terms.} \end{aligned} \qquad (23)$$

The coefficient $V_i(x_0;c_0)=0$, since x_0 is assumed to be a critical point for $c=c_0$. The value of δx is computed by solving $\nabla V(x_0+\delta x; c_0 + \delta c) = 0$. To lowest order (linear), we find

$$V_{ij} \delta x_j + V_{i\alpha} \delta c_\alpha = 0. \qquad (24)$$

If the stability matrix V_{ij} is nonsingular,

$$\partial x_i / \partial c_\alpha = -(V^{-1})_{ij} V_{j\alpha}, \qquad (25)$$

where (V^{-1}) is the matrix inverse of the nonsingular stability matrix: $(V^{-1})_{ij} V_{jk} = \delta_{ik}$. That is, a small change in control-parameter value produces a small change in the location of the critical point, as long as the stability matrix is nonsingular. The matrix (25) is the linear response function for the potential at the equilibrium x_0: it describes how much the equilibrium is displaced by a small change in the control parameters.

To second order the value of the potential at the displaced critical point is

$$\begin{aligned} V(x_0+\delta x;c_0+\delta c) \\ &= V(x_0;c_0) + V_\alpha \delta c_\alpha \\ &\quad + \tfrac{1}{2}[V_{\alpha\beta} - V_{\alpha i}(V^{-1})_{ij} V_{j\beta}] \delta c_\alpha \delta c_\beta. \end{aligned} \qquad (26)$$

The stability matrix at the displaced critical point, $V_{ij}(x_0+\delta x;c_0+\delta c)$, is related to the stability matrix at the original critical point, $V_{ij}(x_0;c_0)$, by

$$V_{ij}(x_0+\delta x;c_0+\delta c) = V_{ij}(x_0;c_0) + P_{ij\alpha} \delta c_\alpha,$$

$$P_{ij\alpha} = V_{ij\alpha}(x_0;c_0) - V_{ijk}(V^{-1})_{kl} V_{l\alpha}. \qquad (27)$$

As a result, for sufficiently small perturbations the Morse saddle type cannot change if the stability matrix is nonsingular.

This application of perturbation theory, and the analytic results constructed in Eqs. (25)–(27), evaporate when the stability matrix V_{ij} becomes singular. Under these conditions the evolution of the dynamical system under change in the control-parameter values ("perestroika") is computed by expressing the potential, $V(x;c)$, in the neighborhood of a non-Morse critical point by an appropriate catastrophe, following the perturbation through the well-defined bifurcation sets, and