SP-345 Evolution of the Solar System

8. RESONANCE STRUCTURE IN THE SOLAR SYSTEM

[123] 8.1. RESONANCES IN THE SOLAR SYSTEM

If we tabulate the orbital and spin periods of all the bodies in the solar system, we find that many of the periods are commensurable, indicating the existence of a number of resonance effects between mutually coupled resonators. There are resonances between the orbital periods of members of the same system and there are also resonances between the orbital and spin periods of rotating bodies.

Such resonances seem to be very important features of the solar system. As bodies once trapped in a resonance may under certain circumstances remain trapped indefinitely, resonance structures stabilize the solar system for very long periods of time.

A study of the resonance structure within a system may give us relevant information about the evolution of that system. To draw any conclusions in this respect we must clarify how the present resonance structure has been established. Two different mechanisms have been suggested:

(1) The first one, which has been proposed by Goldreich (1965), envisages that bodies were originally produced with no resonance coupling of their spin and orbital periods except those necessarily resulting from a random distribution. A later evolution of the system, mainly by tidal effects, changed the periods in a nonuniform way and resulted in the establishment of resonances.

This theory cannot in any case supply a general explanation of resonances. It is applicable only to the satellite systems and, since the tides produced on the Sun by the planets are totally negligible, another process must be invoked to explain the establishment of resonances in the planetary system. Further, the explanation of resonances as a tidal effect runs into difficulties even when applied only to satellite resonances. For example, as according to sec. 18.6 the Cassini division is genetically connected with Mimas, the orbit of Mimas cannot have changed by more than 1 or 2 percent since hetegonic times. Hence, there is not room for much tidal evolution.

(2) According to the alternative suggestion (Alfvén and Arrhenius, 1973), resonance effects were important in the hetegonic process itself, so that...

[**124**] TABLE 8.1.1. Types of Resonances.

.	Satellite orbit	Planetary orbit	Planetary spin
Satellite orbit	Jovian satellites	(Sun and Jovian satellites 8, 9, 11) **a, b**	Tidal effects Possible effects between Earth

	lo-Europa-Gamymede [b]	(Sun and Phoebe) [a, b]	and the Moon in the past [i]
	Saturnian satellites	(Sun and Moon) [a, b]	
	Mimas-Tethys [b,e]		
	Enceladus-Dione [b]		
	Titan-Hyperion [b, e]		
Planetary orbit		(Jupiter-Saturn) [a, b]	Spin-orbit of Mercury [j]
		Neptune-Pluto [c]	Spin of Venus- orbit of the Earth? [j, k]
		Jupiter-asteroids	
		Trojans [g]	
		Thule [f]	
		Hildas [d]	
		Kirkwood gaps [h]	
		Earth-lvar [b,c]	
		Earth-Toro [l,m,n,o]	
		Venus-Toro [m]	

[a] Parentheses denote a near-commensurability, rather than a captured resonance.
[b] Roy and Ovenden (1954), Goldreich (1965).
[c] Cohen et aL (1967).
[d] Schubart (1968).
[e] Brouwer and Clemence (1961a).
[f] Takenouchi (l962).
[g] Brouwer and Clemence (1961b).
[h] Brouwer (1963)
[i] ch. 26.
[j] Goldreich and Peale (1968).
[k] Dyce and Petteneill (1967).
[l] Danielsson and Ip (1972).
[m] Ip and Mehra (1973).
[n] Williams and Wetherill (1973).
[o] Jamiczek et aL (1972).

.....bodies were preferentially produced in states of resonance with other bodies. Hence, the resonance structure may give us direct information about the hetegonic process.

8.1.1. Different Types of Resonances

In the solar system the following types of resonances (see table 8.1.1) have been observed:

[**125**] (1) *Orbit-orbit resonances*. If two planets or two satellites have orbital periods $\mathbf{T_1}$ and $\mathbf{T_2}$ and the ratio between them can be written

$$\frac{T_1}{T_2} = \frac{n_1}{n_2} \quad (8.1.1)$$

where $\mathbf{n_1}$ and $\mathbf{n_2}$ are small integers, such periods are called commensurable, Resonance effects may be produced if the gravitational attraction between the bodies is above a certain limit. There are several pronounced examples of this in the satellite systems of Jupiter and Saturn, and the effect is also important in the planetary system, especially for the asteroids.

Resonance between the orbital motion of a planet and the orbital motion of one of its own satellites has also been discussed (Roy and Ovenden, 1954). Seen from the frame of reference of the planet, this is a resonance between the apparent motion of the satellite and the apparent motion of the Sun . Such resonances are sometimes referred to as "satellite-Sun resonances."

(2) *Spin-orbit resonances*. If the density distribution in a rotating body is asymmetric, this asymmetry produces a periodically varying gravitation field that may couple with its orbital motion. This effect generally leads to a spin-orbit resonance. The spin of Mercury seems to be locked in a resonance with its own orbital period. Whether the spin of Venus is coupled with the orbital motion of the Earth (in relation to Venus) is a matter of dispute; see sec. 8.8.

A similar asymmetry of a planet may also affect the motion of a satellite revolving around the planet. This effect is not known to be important today but it may have affected the evolution of the Earth-Moon system (ch. 24).

If a satellite produces tides on its primary, the tidal bulges corotate with the satellite. The coupling between the tidal bulges and the satellite may be considered as a spin-orbit resonance with $\mathbf{n_1} = \mathbf{n_2} = 1$.

8.2. RESONANCE AND THE OSCILLATION OF A PENDULUM

In order to study the basic properties of resonances we first treat some simple models.

As pointed out by Brown and Shook (1964), there is a certain similarity between the resonances in the solar system and the motion of a simple pendulum (see fig. 8.2.1). Consider the motion of a mass point **m**, which is confined to a circle with radius 1, under the action of Earth's gravitational....

[**126**]

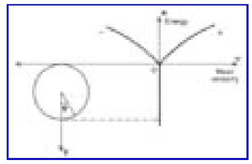

FIGURE 8.2.1. Oscillations of a simple pendulum. If the energy is negative, the pendulum oscillates with an amplitude $\psi_1 < \pi$ and the mean velocity $d\psi/dt$ is zero. If the energy is positive the motion consists of a constant revolution modulated by an oscillation of the same period. The angular velocity ω of this revolution may oe either positive or negative.

....acceleration **g**. If the angle with the vertical is called ψ, the motion is described by the equation

$$\frac{d^2\psi}{dt^2} + A^2 \sin \psi = 0 \quad (8.2.1)$$

where

$$A^2 = \frac{g}{l} \quad (8.2.2)$$

Integrating eq. (8.2.1) we find

$$\left(\frac{d\psi}{dt}\right)^2 = \kappa + 2A^2 \cos \psi \quad (8.2.3)$$

where κ is constant.

Normalizing the energy **W** of the system so that **W** = 0 when the pendulum is at rest at $\psi = \pi$, we have

[**127**]

$$W = \frac{ml^2}{2}\left(\frac{d\psi}{dt}\right)^2 - mgl(1 + \cos \psi) \quad (8.2.4)$$

and from eq. (8.2.3) we find that

$$\kappa = \frac{2W}{ml^2} + 2A^2 \qquad (8.2.5)$$

Depending on the value κ we have three cases:

(1) $\kappa > 2\mathbf{A}^2$; **W** > O. In this case $\frac{d\psi}{dt}$ never vanishes; it could be either > 0 or < 0. We have

$$t - t_0 = \int_{\psi_0}^{\psi} \frac{d\psi}{(\kappa + 2\,A^2 \cos\psi)^{1/2}} \qquad (8.2.6)$$

where $\mathbf{t_0}$ is the value of t when $\psi = \psi_0$. The angle is a constant. If we put

$$\frac{1}{\omega} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{(\kappa + 2\,A^2 \cos\psi)^{1/2}} \qquad (8.2.7)$$

we can write the solution (See Brown and Shook, p.219)

$$\psi - \psi_0 = \omega t + \psi_0 - \frac{A^2}{\omega^2} \sin(\omega t + \psi_0) + \frac{A^4}{8\omega^4} \sin 2(\omega t + \psi_0) + \cdots \qquad (8.2.8)$$

The motion consists of a constant revolution with the period $2\pi/\omega$, superimposed upon an oscillation with the same period. The motion cam proceed in either direction (ω < 0 or ω > 0).

(2) $\kappa < 2\,A^2$; W < O. In this case $\frac{d\psi}{dt}$ = 0 when $\psi = \pm\psi_1$ with

$$\cos\psi_1 = -\frac{\kappa}{2\,A^2} = -\frac{W}{mlg} - 1 \qquad (8.2.9)$$

[**128**] and the integral is

$$\left(\frac{d\psi}{dt}\right)^2 = 4\,A^2\left(\sin^2\frac{\psi_1}{2} - \sin^2\frac{\psi}{2}\right) \qquad (8.2.10)$$

The value of ψ oscillates between $-\psi_1$ and $+\psi_1$. For small amplitudes the period is 2π/**A**; for large amplitudes anharmonic terms make it larger.

(3) The case κ = 2 **A**^{2}; **W** = 0 means that the pendulum reaches the unstable equilibrium at the uppermost point of the circle, with zero velocity. The lowest state of energy occurs when the pendulum is at rest at ψ = 0. If energy is supplied, oscillations start and their amplitude grows until ψ_1 approaches π. Then there is a discontinuous transition from case (2) to case (1).

8.3. A SIMPLE RESONANCE MODEL

In order to demonstrate a basic resonance phenomenon, let us discuss a very simple case (fig. 8.3.1). Suppose that a planet at **O** is encircled by two satellites, one of significant mass ($\mathbf{M_2}$) moving in a circular orbit and one with negligible mass ($\mathbf{M_1}$) moving in an elliptic orbit. We denote by $\omega_1 = 2\pi/T_1$ and $\omega_2 = 2\pi/T_2$ the average angular velocities of $\mathbf{M_1}$ and $\mathbf{M_2}$, and we treat the case where the ratio ω_1/ω_2 is close to 2. Orbital inclinations are put equal to zero.

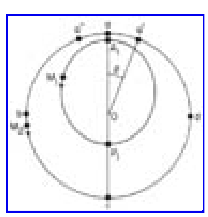

FIGURE 8.3.1.- A planet at **O** has two orbiting satellites, one of small mass ($\mathbf{M_2}$) moving in a circular orbit and one of negligible mass ($\mathbf{M_1}$) moving in an elliptic orbit. The orbital inclinations are zero and the

ratio of average angular velocities ω_1/ω_2 is near 2; $\omega_1 = 2\pi/T_1$ and $\omega_2 = 2\pi/T_2$. $\mathbf{A_1}$ is the apocenter of the inner satellite, and $\mathbf{P_1}$ is the pericenter.

[**129**] If at a certain moment the longitude angles of the satellites are ϕ_1 and ϕ_2 , a "conjunction" occurs when $\phi_1 = \phi_2$. Consider the case when there is a conjunction between the satellites at the moment when the inner one is at its apocenter $\mathbf{A_1}$ and the outer one is at point **a**. This implies that after $\mathbf{M_2}$ completes 1.5 revolutions the outer satellite is at d when the inner satellite is at its pericenter $\mathbf{P_1}$. When $\mathbf{M_1}$ moves from $\mathbf{P_1}$ to $\mathbf{A_1}$ it is subject to the attraction from $\mathbf{M_2}$ which works in the direction of motion, hence increasing the angular momentum. When the motion continues from $\mathbf{A_1}$ to $\mathbf{P_1}$, $\mathbf{M_1}$ is subject to a similar force from the outer satellite, which moves from **a** to **b**, but this force will diminish the angular momentum of $\mathbf{M_1}$. Because of the symmetry the net result is zero (neglecting high-order terms).

Suppose now that $\mathbf{M_1}$ arrives at $\mathbf{A_1}$ a certain time Δ**t** before $\mathbf{M_2}$ arrives at **a**. Because the orbits are closest together around $\mathbf{A_1 a}$, the effects in this region predominate. If $\mathbf{M_2}$ is at **a'** when $\mathbf{M_1}$ is at $\mathbf{A_1}$, the force between them will decrease the angular momentum $\mathbf{C_1}$ of M_1. (The reciprocal effect on $\mathbf{M_2}$ is negligible because of the smallness of $\mathbf{M_1}$.) As the orbital period of a satellite is proportional to $\mathbf{C^3}$, the period of $\mathbf{M_1}$ will be shortened with the result that at the next conjunction it will arrive at $\mathbf{A_1}$ when $\mathbf{M_2}$ is still further away from **a**. The result is that the angle Θ between the bodies when $\mathbf{M_1}$ is at its pericenter will increase.

If on the other hand Θ is negative so that $\mathbf{M_1}$ arrives too late at $\mathbf{A_1}$, say when $\mathbf{M_2}$ already has reached **a''**, the angular momentum of $\mathbf{M_1}$ will increase with the result that Θ will become still more negative.

We can compare this result with the pendulum treated in sec. 8.2 when it is close to the upper point $\psi = \pi$. Putting $\theta = \pi - \psi$ we see that the conjunction at $\Theta = 0$ represents an unstable equilibrium. We can conclude that a stable equilibrium is reached when $\Theta = \pi$, corresponding to $\psi = 0$. This means that the inner satellite is at $\mathbf{P_1}$ when the outer one is at **a**. (This implies that $\mathbf{M_1}$ also is at $\mathbf{P_1}$ when $\mathbf{M_2}$ is at **c**. The interaction at this configuration is smaller than near $\mathbf{A_1}$ because of the larger distance between the orbits.)

8.4. DEVIATIONS FROM EXACT RESONANCE

If we put the mean longitudes of the two bodies equal to ϕ_1 (**t**) and ϕ_2 (**t**), resonance implies that ϕ_1 and ϕ_2 increase such that the average value of the libration angle ξ

$$\langle \xi \rangle = n_1\phi_1 - n_2\phi_2 \quad (8.4.1)$$

is zero.

[**130**] The bodies can oscillate around the equilibrium position. (In celestial mechanics the word 'libration' is used for oscillations.) Libration implies that ϕ_1 and ϕ_2 increase such that ξ varies periodically with a period that may be many orders of magnitude larger than the orbital period. This corresponds to the oscillations of the simple pendulum in case (2).

In the cases we will discuss, the equilibrium position of a body (body 1) in relation to the orbital pattern of body 2 is at $\mathbf{A_1}$, which is located on the apsis line (joining the apocenter and the pericenter). However, the time $\mathbf{T_2}$ needed for body 2 to move one turn in relation to the apsis line is not the sidereal period $\mathbf{T_K}$ because of the precession of the perihelion with the angular velocity ω_P (ch. 3). According to eq. (3.3.12) we have:

$$\omega_2 = \omega_K - \omega_P \quad (8.4.2)$$

with $\omega_K = 2\pi/T_K$ and $\omega_2 = 2\pi/T_2$. Putting ω_1 we find from eq. (8.1.1)

$$\omega_1 = \frac{n_2}{n_1}\omega_2 = \frac{n_2}{n_1}(\omega_K - \omega_P) \quad (8.4.3)$$

Furthermore, in case of libration body 1 is not situated at $\mathbf{A_1}$ but at an angle ξ (**t**) from it. During one period $\mathbf{T_1}$, the angle changes by $T_1\frac{d\xi}{dt}$. From eq. (8.4.1) we find

$$\left\langle \frac{d\xi}{dt} \right\rangle = n_1\omega_1 - n_2(\omega_K - \omega_P) \quad (8.4.4)$$

If eq. (8.4.3) is satisfied, there is a coupling between perihelion position and the resonant orbital coupling of the bodies; the average value of the libration angle ξ is constant, and eq. (8.4.4) reduces to zero.

The amplitude of the libration is a measure of the stability of the resonance coupling. If the amplitude of the libration is increased to π the system passes discontinuously from a state of finite amplitude libration (case (2)) to a state of revolution modulated by periodic oscillation (case (1)). In the latter state, the resonance is broken but a "near-commensurability" exists, and the average value of ξ for the system will increase or decrease indefinitely with time.

[131] 8.5. ORBIT-ORBIT RESONANCES

To study the resonance phenomena in the solar system, one can start from the equations of motion of a pendulum disturbed by a periodic force (Brown and Shook, 1964). The problems usually lead to analytically complicated formulae that can be treated only by elaborate computer calculations. Very often only numerical solutions of a number of typical cases can clarify the situation. It is beyond the scope of our treatise to discuss this in detail. Instead we shall treat some simple cases that demonstrate the basic physical phenomena.

In the solar system there are a number of orbit-orbit resonances; i.e., resonances between satellites (or planets) whose motions are coupled in such a way that their orbital periods are commensurate. In this section we shall discuss some of these resonances.

In most cases of resonance the bigger of the two bodies moves in an orbit with very low eccentricity, whereas the orbit of the small body has a rather high eccentricity. We can account for essential properties of the resonance phenomena if we approximate the orbit of the more massive body as a circle. Further, we will in general only deal with the case of coplanar orbits.

8.5.1. Neptune-Pluto

One example of an orbit-orbit resonance is the Neptune-Pluto system, which has been studied by Cohen and Hubbard (1965), who have integrated the orbits over an interval of 10^6 yr. Their results were later essentially confirmed by Williams and Benson (1971), whose integrations cover 4.5 X 10^6 yr. The orbital periods of Neptune and Pluto are T_{Ψ} = 165 yr and T_{P} = 248 yr, which from eq. (8.1.1) gives $\mathbf{n_1}$ = 2 and $\mathbf{n_2}$ = 3. Figure 8.5.1 shows the orbit of Pluto (as found by numerical integration) in a reference system where the Sun and Neptune are at rest. In this system it takes Pluto 500 yr ($T_{\Psi}\ T_{\text{P}}\ /\ T_{\text{P}}\ -\ T_{\Psi}$) to complete one turn.

In relation to the Plutonian orbit, Neptune may be located at any point of the arc bac. If it is located in the middle (at **a**), its gravitational attraction on Pluto integrated over am entire 500-yr period is zero because of the symmetry. If Neptune is located at **b**, its gravitational attraction will be stronger on the left part of the Plutonian orbit, with the result that orbital angular momentum will be transferred from Neptune to Pluto. This transfer will increase the orbital period of Pluto and reduce the period of Neptune. The result is that, in relation to the orbital pattern of Pluto, Neptune will begin to move toward the right along the arc. We can express the result by saying that, if Neptune is placed at **b**, it will *appear to be repelled* by the closeness of the Plutonian orbit. Similarly, if Neptune is located at **c**, it will appear to be repelled toward the left due to the closeness of the orbit of Pluto.

[**132**]

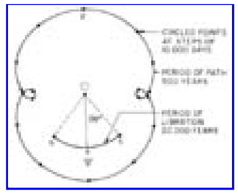

FIGURE 8.51.- The orbit of Pluto with respect to the Sun and to Neptune. The orbital pattern of Pluto librates relative to Neptune, but for clarity the Plutonian orbit is held stationary and the libration of Neptune relative to it is shown. The equilibrium position of Neptune is at a and Neptune librates between extreme positions at b and c with an amplitude of 38°. (From Cohen and Hubbard, 1965.)

Hence, in relation to the Plutonian orbit, Neptune will oscillate between **b** and **c**, in a way similar to the pendulum in fig. 8.2.1. Cohen and Hubbard (1965) have found the period of this libration to be about 20 000 yr. The double amplitude of libration is 76°. The minimum distance between Pluto and Neptune is 18 AU. Hence, because of the resonance, Neptune and Pluto *can never collide* in spite of the fact that these orbits intersect.

The period covered by numerical integrations is only 10^{-3} of the age of the solar system, so it is dangerous to extrapolate back in time to the hetegonic era. It seems unlikely that gravitational effects alone could have changed the amplitude of libration so much that a resonance capture will be found to have occurred long ago. However, viscous forces from a surrounding dispersed medium could, of course, have produced such a change. Such a process would necessarily have led to an appreciable accretion of this medium by Pluto. This means that the establishment of the resonance is likely to be connected with the general problem of planetary accretion. Hence, we tentatively conclude that *the present pattern is likely to have been established as a result of hetegonic processes*. Thus, by studying this and other resonances we may get important information about the hetegonic processes.

Lyttleton (1936), Kuiper (1957), and Rabe (1957a and b) have suggested that Pluto might be a runaway satellite of Neptune. This idea was put forward before the resonance was discovered and now seems very unlikely [**133**] because there is no obvious mechanism consistent with this idea that can account for the establishment of the resonance. In spite of that, the idea appears to still be frequently quoted.

8.5.2. Earth-Toro and Other Earth-Asteroid Resonances

As has been discovered recently (Danielsson and Ip, 1972), the Earth and Toro form an 8/5 resonance system (fig. 8.5.2). In a Sun-Earth frame of reference, Toro makes five loops similar to the two orbital loops of Pluto The Earth oscillates on the arc bac, being apparently repelled whenever it comes close to Toro's orbit. In contrast to the Neptune-Pluto resonance, the resonance capture is established by two very close encounter taking place during two rapid passages in an 8-yr period. During the rest of the 8-yr period, the interaction is almost negligible.

If the encounters with the Earth were the only close encounters, the Earth-Toro pattern would have a permanent life. However, Toro's motion is complicated by the fact that its perihelion is close to Venus' orbit. The....

FIGURE 8.5.2.- Projection of I685 Toro on the ecliptic plane in a coordinate system rotating with the Earth. Between 1600 AD and 1800 AD, the Earth-Sun Iine librates in the **b'a'c'** domain about the equilibrium position **a'**. The libration makes the transit to the bac domain around 1850 AD and remains there until 2200 AD. After 2200 AD the Earth-Sun equilibrium position will shift back from **a** to **a'**. The orbital pattern of Toro librates relative to the Earth, but for the sake of clarity the Earth is depicted as librating in relation to the orbital pattern of Toro. (From Ip and Mehra, 1973.)

[**134**] ....result is that close encounters between Venus and Toro periodically shift the Earth-Toro pattern so that the Earth for a certain period oscillates along the arc **b'a'c'**. A subsequent encounter with Venus brings it back again. The crossings are possible because the orbital planes differ.

As has been shown by Danielsson and Mehra (1973), this periodic shift between two capture positions might have been permanent if only Toro, the Earth, and Venus had been involved. However, the aphelion of Toro is outside the orbit of Mars, and, as pointed out also by Williams and Wetherill (1973), it seems that close encounters with Mars are statistically probable and will make the resonance transitory with a duration much smaller than the age of the solar system. It seems at present impossible to reconstruct the orbit of Toro back to hetegonic times.

There are a number of other asteroids which are in resonance capture of a more or less permanent character. Surveys are given by Janiczek et al. (1972), Ip and Mehra (1973), and Danielsson and Mehra (1973). Ivar is trapped in a 11/28 resonance, which probably is rather stable, and Amor is trapped in a 3/8 resonance, which is unstable.

8.5.3. The Trojans

The Trojans are in a 1/1 resonance with Jupiter. They librate around the Lagrangian points of Jupiter. Figure 8.5.3. shows regions within which the librating Trojans are confined. Due to the eccentricity of Jupiter's orbit and perturbations from other planets, the three-dimensional motions of the Trojans are extremely complicated, having several different libration periods (Brouwer and Clemence, 1961b). Whether, in some cases, these librations may be so large as to throw some Trojans out of libration is still undetermined.

As the outermost Jovian satellites have a retrograde motion, they must have been gravitationally captured. It seems reasonable that there is a connection between these satellites and the Trojans, and it is possible that the satellites are captured Trojans. Whether this capture has taken place under present conditions or during the hetegonic era is still to be clarified.

8.5.4. The Hilda Asteroids

The Hilda asteroids, named after the biggest member of the group, are in 2/3 resonance with Jupiter. These asteroids have been studied by Chebotarev (1967) and Schubart (1968). Approximating Jupiter's motion as circular, and neglecting the inclinations between the orbits, the motion of a typical Hilda asteroid is shown in fig. 8.5.4.

The resonance mechanism can be explained in the same simple way as in the earlier cases: As soon as Jupiter comes close to the orbital pattern.....

[**135**]

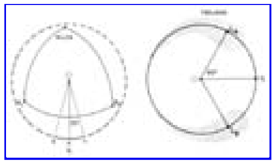

FIGURE 8.5.4.- Idealized orbit of 153 Hilda in the Jupiter-Sun rotating coordinate system. Due to the 3/2 resonance, Hilda describes a triangular trajectory in a time interval of 24 yr. Due to systematic perturbations the whole orbital pattern will oscillate with an amplitude of 15° and a period of 260 yr. Points $\mathbf{A_1}$ and $\mathbf{A_2}$ are the aphelia of Hilda and also her points of closest approach to Jupiter. The distance between Hilda and Jupiter at close approach is never less than 4 AU. (From Ip, 1974a.)

...of the asteroid, there is an apparent repulsion. Hence, the equilibrium position is at **a**, but normally there are librations for example between **b** and **c**.

In the cases earlier discussed, the orbits of the two bodies in resonance crossed each other. This means that in the planar case there is no possibility to establish or break the resonance without a close encounter between the bodies. If the orbital planes do not coincide, the situation is more complex.

The orbits of the Hildas do not cross the orbit of Jupiter; therefore, a continuous transition to a nonresonant case is possible. An increase in the amplitude of the oscillations may eventually result in a transition to the noncaptive state, such that Jupiter (fig. 8.5.4) begins to librate in relation to the orbital pattern in the same way as the pendulum in fig. 8.2.1. does for the case **W** > 0.

[**136**] The asteroid Thule is also resonance-captured by Jupiter (ratio 3/4). Its librations have been studied by Takenouchi (1962) and by Marsden (1970). The resonances Jupiter-Hildas and Jupiter-Thule are of importance in the discussion of the Kirkwood gaps (secs. 4.3 and 8.6). It is evident that there are clusters of bodies at Jovian resonance points, and the theoretical studies show that there are good reasons for this. This indicates that the Kirkwood gaps (absence of bodies at Jovian resonance points) cannot simply be resonance phenomena but are due to other factors; e.g., collision phenomena (Jefferys, 1967; Sinclair, 1969).

8.5.5. Titan-Hyperion

In the Saturnian system the small satellite Hyperion moves in an eccentric orbit outside Titan (fig. 8.5.5). The equilibrium position is reached at conjunction when Hyperion is at its aposaturnian. For further details see Roy and Ovenden (1954), Goldreich (1965), and Brouwer and Clemence (1961a).

8.5.6. Dione-Enceladus

This resonance, also in the Saturnian system, is of the type 1/2. The pattern is shown in fig. 8.5.6; as the libration of Enceladus is only 11 sec of arc, it is not shown. The orbit of Dione is approximated by a circle, and.....

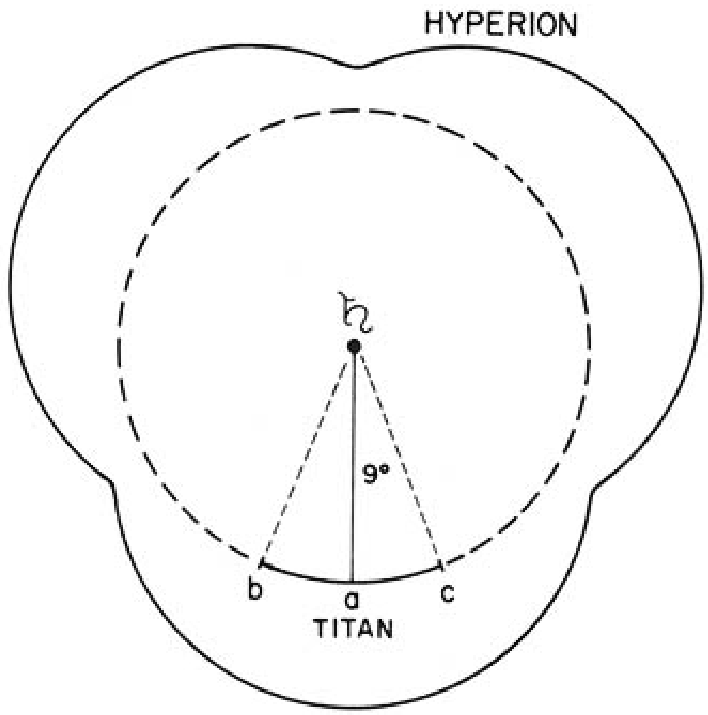

FIGURE 8.5.5.- The orbital pattern for the 4/3 resonance of Titan-Hyperion in the Saturnian satellite system. Titan librates with an amplitude of 9° about the equilibrium position at **a**. The orbit of Hyperion is strongly perturbed by Titan.

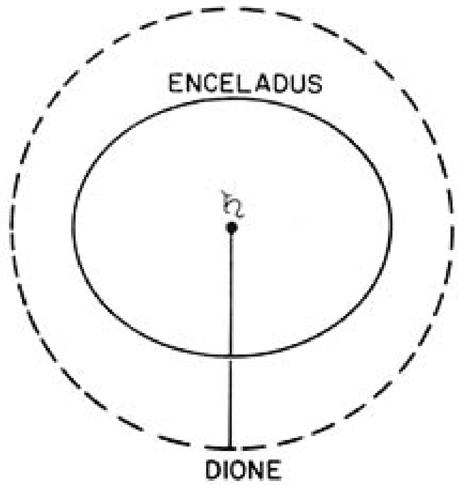

FIGURE 8.5.6.- The orbital pattern of the 2/1 resonance of Enceladus-Dione in the Saturnian system. The ellipticity of the orbit of Enceladus is exaggerated for the purpose of illustration. The perisaturnian of Enceladus precesses due to Dione.

....the eccentricity of Enceladus' orbit is exaggerated for the sake of clarity. This resonance is discussed at greater length by Roy and Ovenden (1954).

8.5.7. Tethys-Mimas

It should be pointed out that if the orbits are coplanar a prerequisite for resonance is that at least one of the orbits is eccentric. If both orbits are exactly circular, no coupling between the orbits is produced.

In all the preceding cases am approximation to coplanar motion illustrated the essential character of the resonance. In contrast, the resonance between the Saturnian satellites Tethys and Mimas is dependent on the inclination of the orbits, and the resonance is related to the nodes. This is also the case for the Jupiter-Thule resonance. Detailed discussions are given by Roy and Ovenden (1954), Goldreich (1965), and Brouwer and Clemence (1961a).

8.5.8. Io-Europa-Ganymede

A more complicated case of commensurability is found in the Jovian system, where the angular velocities of Io, Europa, and Ganymede obey the relationship

$$\omega_I - 3\omega_E + 2\omega_G = 0 \quad (8.5.1)$$

to within the observational accuracy 10^{-9}. The mechanism is rather complicated. It has been treated in detail by the exact methods of celestial mechanics; e.g., Roy and Ovenden (1954).

Table 8.5.1 gives a survey of all known orbit-orbit resonances.

[**138**] TABLE 8.5.1. Orbit-Orbit Resonances in the Solar System

Bodies	Orbital Parameters		
	e	**i** (°)	Period (da/yr)
Tethys	0.00	1.1	1.887802
Mimas	0.0201	1.5	0.942422
Dione	0.0021	0.0	2.73681
Enceladus	0.0045	0.0	1.37028
Hyperion	0.104	0.5	21.27666
Titan	0.0290	0.3	15.945452
Pluto	0.247	17.1	248.43
Neptune	0.0087	1.46	164.78
Jupiter	0.048	1.38	11.86
Hilda	0.15	7.85	7.90
Jupiter	0.048	1.38	11.86
Thule	0.03	23.	8.90
Jupiter	0.048	1.38	11.86
Trojans	~0.15	10-20	11.86
Earth	0.017	0.0	1.0
Toro	0.435	9.3	1.6
Earth	0.017	0.0	1.0
Ivar	0.397	8.3	2.545

[**139**] TABLE 8.5.1. Orbit-Orbit Resonances in the Solar System (Continued)

Ratio	Resonance Type	Libration		References
		Period (yr)	Amplitude (°)	
1	Resonances related to the nodes	70.8	47	(a)

2	.			(b)
.				
1	Conjunction when Enceladus at peri-saturnian	3.89	11'24"	(a)
2	.			
.				
3	Conjunction when Hyperion at apo-saturnian	18.75	9	(a)
4	.			(b)
.				
2	See fig. 8.5.1	20 000	39	(c)
3	.			
.				
2	Largest body of a group of at least 20 bodies librating with different amplitudes and phases	270	40	(d)
3	.			
.				
3	.	500	~0	(e)
4	.			(f)
.				
1	Two groups, one at each of the libration points of Jupiter	~900	10-20	(g)
1	.			
.				
8	Resonance due to close encounter	150	10	(h)
5	.			(i)
.				
28	Resonance due to close encounter	300	26	(i)
11	.			

a. Roy and Ovenden (1954), Goldreich (1965).
b. Brouwer and Clemence (1961n).
c. Cohen et al. (1967).
d. Schubart (1968).
e. Takenouchi (1962).
f. Ip (1974a)
g. Brouwer and Clemence (1961b).
h. Danielsson and Ip (1972).
i. Ip and Mehra (1973).

[**140**] 8.6. THE KIRKWOOD GAPS

An interesting and puzzling resonance-related phenomenon is found in the main asteroidal belt (see fig.

4.3.3). If the number of asteroids is plotted as a function of orbital period, or equivalently as a function of semimajor axis, there are a number of pronounced empty zones, the so-called Kirkwood gaps, in the neighborhood of periods commensurable with Jupiter's. Gaps corresponding to resonances of 1/2, 1/3, 2/5, and 3/7 are clearly observed and some higher resonances have also been suggested (see sec. 4.3).

The Kirkwood gaps have attracted much interest, and there is a multitude of theoretical papers about the mechanism producing them (Brouwer, 1963; Schweizer, 1969; Sinclair, 1969). Some of the authors claim to have made theoretical models that adequately explain the gaps. If one tries to extract the fundamental physical principles of these models from the jungle of sophisticated mathematical formulae, one does not feel convinced of the explanations. Doubt of the adequacy of these models is aroused by the fact that, whereas both Tethys and Dione are keeping small bodies (Mimas and Enceladus) trapped at resonance 1/2, Jupiter *produces an absence* of small bodies at the corresponding period. Further, Jupiter keeps a number of Hilda asteroids trapped in a 2/3 resonance but produces gaps at a number of other resonance points in the main asteroid belt. It is essential that any theory of the Kirkwood gaps simultaneously explain both types of resonance phenomena.

In the absence of a clear answer to these questions, one must ask whether the Kirkwood gaps really are produced by the resonance effects of the type discussed by the current theories. As we have seen in ch. 5, there are reasons to believe that nongravitational effects are of importance to the motion of comets and asteroids. It is therefore possible that Jefferys (1967) is correct when he suggests that nongravitational effects (e.g., collisions) are essential for an understanding of the Kirkwood gaps. If the gaps were the result of a hetegonic process, this would make them more interesting from the point of view of the early history of the solar system. One hopes that a complete theory of the formation of the asteroid belt will afford a thorough explanation of the Kirkwood gaps.

8.7. ON THE ABSENCE OF RESONANCE EFFECTS IN THE SATURNIAN RING SYSTEM

The dark markings in the Saturnian ring system, especially Cassini's division, have long been thought to be due to resonances produced by Mimas and perhaps by other satellites as well. It has been claimed that the gaps in the Saturnian rings ought to be analogous to the Kirkwood gaps in the [**141**] asteroid belt. Such an analogy is erroneous because it has been shown both observationally and theoretically (see Alfvén, 1968) that the Saturnian rings cannot be explained as a resonance phenomenon.

The accurate measurements of Dollfus (1961) are shown in fig. 18.6.1. It is obvious that there is no acceptable correlation between the observed markings and such resonance-produced gaps as would be expected in analogy to the Kirkwood gaps in the asteroid belt. Furthermore, the mass ratio of Mimas to Saturn is $1/(8 \times 10^6)$, whereas the mass ratio of Jupiter to Sun is $1/10^3$. Hence, the relative perturbation effect is 10' times smaller in the case of the Saturnian rings than in the case of Jupiter and the asteroid belt. Such a small gravitational perturbation is not likely to produce any appreciable resonance phenomenon.

As we shall see in sec. 18.6, the dark markings are readily explainable as hetegonic "shadow" effects.

8.8. SPIN -ORBIT RESONANCES

For all satellites with known spins the spin periods equal the orbital periods. This is likely to be due to tides, produced by their primaries, which have braked the synodic rotations of the satellites to zero. For a formal statement of such a resonance we have

$$\frac{\tau}{T} = \frac{n_\tau}{n_T} \quad (8.8.1)$$

where **T** is the orbital and τ is the spin period of the body in question, and $n_\tau = n_T = 1$.

Mercury's spin period is 59 days, which is exactly 2/3 of its orbital period (Dyce and Pettengill, 1967). This means that Mercury is captured in a spin-orbit resonance. According to Goldreich and Peale (1968), this represents the final state produced by the solar tide.

The case of Venus is puzzling. It has a retrograde spin with a period of about 243 days. The spin period of Venus is supposedly in a 5/4 resonance with the orbital period of the Earth as seen from Venus (Dyce and Pettengill, 1967; Goldreich and Peale, 1968). It is surprising that the Earth can lock Venus into such a resonance (Kaula, 1968). New measurements seem to cast doubt on the reality of this resonance (Carpenter, 1970).

Another type of spin-orbit resonance is that of a spinning body such as a planet and the satellites around it. Allan (1967) has drawn attention to the fact that, if the gravitational potential of the planet depends on the longitude, a satellite will be subject to a force in the tangential direction [**142**] that may transfer energy between the planetary spin and the orbiting satellite. In case the orbiting period of the satellite equals the spin period of the planet, we have a 1/1 resonance. The satellite will be locked at a certain phase angle around which it can librate. There are no examples of synchronous natural satellites, but the theory is applicable to geostationary artificial satellites.

There are also higher resonances ($\mathbf{n_T}$ and n_τ take on larger values), but these are efficient only for satellites with high inclinations or high eccentricities. A body in a circular orbit in the *equatorial* plane is not affected. It has been suggested that such resonances were of importance during the evolution of the Earth-Moon system (see ch. 24).

8.9. NEAR-COMMENSURABILITIES

Besides the exact resonances there are a number of near-commensurabilities. In the development of celestial mechanics such near-commensurabilities have attracted much attention because the perturbations become especially large. Most noteworthy is the case of Jupiter-Saturn, whose periods have a ratio close to 2/5. The near-commensurabilities have been listed by Roy and Ovenden (1954) and further discussed by Goldreich (1965).

In the case of exact resonances, the relative positions of the bodies are locked at certain equilibrium positions around which they perform oscillations as shown in figs. 8.5.1 through 8.5.6. At near-commensurability no such locking exists. In relation to the orbital pattern of body 2, body 1 continuously revolves, just as the pendulum (fig. 8.2.1) in case (1). It is possible that some or all of these near-commensurabilities are broken captured resonances. This would be likely if the hetegonic processes had a strong preference for generating bodies in resonance. However, so far it is doubtful whether near-commensurabilities really are of hetegonic significance. If the periods of the different bodies are distributed at random, there is a certain probability that two periods should be near-commensurable. Studies by the authors cited agree that the number of observed commensurabilities is larger than expected statistically. If, however, we account for the exact resonances by a separate mechanism and subtract them, the remaining statistical excess, if any, is not very large.

Of interest from a hetegonic point of view are the near-commensurabilities of retrograde satellites and the Sun (Roy and Ovenden, 1954). The Jovian satellites 8, 9, and 11 have periods that are close to 1:6 of the orbital period of Jupiter; for 12 the ratio is close to 1:7. The same is the case for the period of Phoebe compared to the period of Saturn. There is a possibility that these commensurabilities were significant for the capture of these satellites ("resonance capture").

[143] 8.9.1. Transition From Capture to Near-Commensurability

There are two basic ways in which a capture resonance can be broken.

(1) The libration may increase. In the case of the pendulum, this corresponds to an increase in energy so that **W** passes from < 0 to > 0.

(2) A torque is applied that is stronger than the resonance can tolerate.

To take the simpler case of applied torque, suppose that the librations are zero. If we apply a torque to the pendulum, it will be deviated am angle ψ from its equilibrium. With increasing torque ψ will increase. When it reaches the value π/2, the restoring force begins to decrease. Hence, if the torque exceeds the value corresponding to $\psi = \pi/2$ the pendulum starts a continuous accelerated motion, and the capture is broken.

To apply this result to the celestial problem, suppose that two celestial bodies are captured in resonance and one of them is subject to a drag; e.g., from the Poynting-Robertson effect. The angle ψ will increase, and the drag will be compensated by the resonance force. If a certain maximum permissible drag is exceeded, the capture will be broken. In relation to the orbital period of body 2, body 1 will begin to revolve, and a near-commensurability will be established.

INDEX PAGE