POSSIBLE NEW EFFECTS IN SUPERCONDUCTIVE TUNNELLING *

B. D. JOSEPHSON
Cavendish Laboratory, Cambridge, England

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We here present an approach to the calculation of tunnelling currents between two metals that is sufficiently general to deal with the case when both metals are superconducting. In that case new effects are predicted, due to the possibility that electron pairs may tunnel through the barrier leaving the quasi-particle distribution unchanged.

Our procedure, following that of Cohen et al. [1], is to treat the term in the Hamiltonian which transfers electrons across the barrier as a perturbation. We assume that in the absence of the transfer term there exist quasi-particle operators of definite energies, whose corresponding number operators are constant.

A difficulty, due to the fact that we have a system containing two disjoint superconducting regions, arises if we try to describe quasi-particles by the usual Bogoliubov operators [2]. This is because states defined as eigenfunctions of the Bogoliubov quasi-particle number operators contain phase-coherent superpositions of states with the same total number of electrons but different numbers in the two regions. However, if the regions are independent these states must be capable of superposition with arbitrary phases. On switching-on the transfer term the particular phases chosen will affect the predicted tunnelling current. This behaviour is of fundamental importance to the argument that follows. The neglect, in the quasi-particle approximation, of the collective excitations of zero energy [3] results in an unphysical restriction in the free choice of phases, but may be avoided by working with the projected states with definite numbers of electrons on both sides of the barrier. Corresponding to these projections we use operators which alter the numbers of electrons on the two sides by definite numbers **. In particular, corresponding to the Bogoliubov operators α_k^+ we use quasi-particle creation operators α_{ek}^+, α_{hk}^+ which respectively add or remove an electron from the same side as the quasi-particle and leave the number on the other side unchanged, and pair creation operators S_k^+ † which add a pair of electrons on one side leaving the quasi-particle distribution unchanged. The Hermitean conjugate destruction operators have similar definitions. The S operators, referring to macroscopically occupied states, may be treated as time dependent c-numbers ††, and we normalise them to have unit amplitude. Relations expressing electron operators in terms of quasi-particle operators, equal-time anticommutation relations and number operator relations may be derived from those of the Bogoliubov theory by requiring both sides of the equations to have the same effect on N_l and N_r, the numbers of electrons on the two sides of the barrier. For example,

$$a_{k\uparrow}^+ = u_k\, \alpha_{ek0}^+ + v_k\, \alpha_{hk1}\,, \qquad \alpha_{ek}^+\, \alpha_{ek} = n_k\,, \qquad \alpha_{ek}^+\, \alpha_{hk} = S_k^+\, n_k\,. \tag{1}$$

Noting that the Bogoliubov Hamiltonian is $H - \lambda N$ (λ = chemical potential), we take our unperturbed Hamiltonian to be

$$H_0 = \sum_k n_k E_k + \lambda_l N_l + \lambda_r N_r\,,$$

where E_k is the quasi-particle energy in the Bogoliubov theory, and derive the commutation relations

$$[H_0, \alpha_{ek}^+] = (E_k + \lambda_k)\, \alpha_{ek}^+\,,$$
$$[H_0, \alpha_{hk}^+] = (E_k - \lambda_k)\, \alpha_{hk}^+\,, \tag{2}$$
$$[H_0, S_k^+] = 2\,\lambda_k\, S_k^+\,.$$

In the presence of tunnelling the Hamiltonian is $H_0 + H_T$, where H_T expressed in electron operators is

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** We shall use subscripts l and r to distinguish operators on the two sides, and k to denote an operator referring to either side.

† These are equivalent to the operators which change $|N\rangle$ to $|N+2\rangle$ in the theory of Gor'kov [4].

†† Cf. N. N. Bogoliubov et al. [5]. The phase of an S operator is related to the orientation of the plane containing the pseudospin operators [3]. Physical observables cannot depend on the phase of a single S operator, but they can depend on the relative phases of the S operators associated with two superconducting regions, as in the phenomena dealt with here.

$$\sum_{l,r}^{\rightarrow} (T_{lr}\, a_l^+\, a_r + T_{rl}\, a_r^+\, a_l) \,.$$

If we describe the time dependence of operators by the interaction picture [6], equations (2) imply that the a and S operators have exponential time dependence. The current operator in the Heisenberg picture is related to that in the interaction picture according to

$$J_H(t) = U^{-1}(t)\, J_{\text{int}}(t)\, U(t) \,,$$

where

$$U(t) = \lim_{\varepsilon \to 0+} \{T \exp(-\mathrm{i}/\hbar \int_{-\infty}^{t} e^{\varepsilon t'} H_T(t')\, \mathrm{d}t')\} \,.$$

Here H_T is expressed in the interaction picture and $U(t)$ can be evaluated by writing H_T in terms of quasi-particle operators and using the method of Goldstone [6]. We also express

$$J_{\text{int}}(t) = \mathrm{i}e/\hbar\, [H_T, N_r] \quad 1)$$

in terms of quasi-particle operators, and by retaining only those terms in $J_H(t)$ which can be expressed in accordance with (1) as products of S and number operators obtain an expression equivalent to the usual one, of the form

$$J_H = J_0 + \tfrac{1}{2}\, J_1\, S_l^+\, S_r + \tfrac{1}{2}\, J_1^*\, S_r^+\, S_l \,. \qquad (3)$$

To second order in H_T, J_0 is similar to the expression of Cohen et al. [1], and reduces for the same reasons to the usual one obtained by neglecting coherence factors. The remaining terms oscillate with frequency $\nu = 2eV/h$ ($V = \lambda_l - \lambda_r$ being the applied voltage), owing to the time dependence of the S operators. J_1 is proportional to the effective matrix element for the transfer of electron pairs across the barrier without affecting the quasi-particle distribution, and typical terms are of the form

$$\begin{aligned} &2\,\mathrm{i}\, e\, u_l\, v_l\, u_r\, v_r\, T_{lr}\, T_{-l,-r}\, [(1 - n_{l0} - n_{r1}) \\ &\quad \times \{P \frac{1}{eV - E_l - E_r} + \pi \mathrm{i}\, \delta(eV - E_l - E_r)\} \\ &- (n_{l0} - n_{r0})\, \{P \frac{1}{eV + E_l - E_r} + \pi \mathrm{i}\, \delta(eV + E_l - E_r)\}] \end{aligned} \qquad (4)$$

where $-k$ denotes the state paired with k. The second and fourth terms result from processes involving real intermediate states, and can be regarded as fluctuations in the normal current due to coherence effects. We note that the first term remains finite at zero temperature and zero applied voltage. From (3) our theory predicts that

(i) At finite voltages the usual DC current occurs, but there is also an AC supercurrent of amplitude $|J_1|$ and frequency $2\,eV/h$ (1 μV corresponds to 483.6 Mc/s).

(ii) at zero voltage, J_0 is zero, but a DC supercurrent of up to a maximum of $|J_1|$ can occur.

Applied r.f. fields can be treated by noting that the oscillations in V frequency-modulate the supercurrent. Thus if a DC voltage V on which is superimposed an AC voltage of frequency ν is applied across the barrier, the current has Fourier components at frequencies $2eV/h \pm n\nu$, where n is an integer. If for some n, $2eV/h = n\nu$, the supercurrent has a DC component dependent on the magnitude and phase of the AC voltage. Hence the DC characteristic has a zero slope resistance part over a range of current dependent on the magnitude of the AC voltage.

Equivalent quantum-mechanical explanations of these effects can be given. For example (i) is due to the transfer of an electron pair across the barrier with photon emission, leaving the quasi-particle distribution unchanged. Consequently the photon frequency is not broadened by the finite quasi-particle lifetimes occurring in real superconductors. (ii) is due to pair transfer without photon emission. The linear dependence of the current on the matrix element is due to the fact that the process involves macroscopically occupied states between which phase relationships can occur.

The possibility of observing these effects depends on the value of $|J_1|$. At low temperatures and voltages the first term of (4) dominates, and in the presence of time-reversal symmetry all contributions to it are in phase. $|J_1|$ is then equal to the current flowing in the normal state at an applied voltage equal to π times the energy gap, assumed to be the same on both sides. At higher temperatures the third term reduces $|J_1|$, and at high frequencies the effects are reduced by the capacitance across the barrier. Magnetic fields, and currents in the films destroy the time-reversal symmetry and reduce $|J_1|$. The effects may be taken into account approximately by replacing (3) by

$$j = j_0 + \tfrac{1}{2}\, j_1\, \psi_l^*\, \psi_r + \tfrac{1}{2}\, j_1^*\, \psi_r^*\, \psi_l \,,$$

where j is the tunnelling current density, and ψ_l, ψ_r the effective superconducting wave functions [7] in the films on the two sides. This formula predicts that in very weak fields diamagnetic currents will screen the field from the space between the films, but with a large penetration depth owing to the smallness of j_1. In larger fields, owing to the existence of a critical current density, screening will not occur; the phases of the supercurrents will vary rapidly over the barrier, causing the maximum total supercurrent to drop off rapidly with increasing field. Anderson [8] has suggested that the absence of tunnelling supercurrents in most experiments hitherto performed may be due to the earth's field acting in this way. Cancellation of supercurrents would start to occur when the amount of flux between the films, including that in the penetration regions, became of the order of a quantum of flux $hc/2e$. This would occur for typical films in a field of about 0.1 gauss. Such a field would not be appreciably excluded by the critical currents obtainable in specimens of all but the highest conductivity.

When two superconducting regions are separated by a thin normal region, effects similar to those considered here should occur and may be relevant to the theory of the intermediate state.

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EXPANSION OF THE SCATTERING AMPLITUDE IN RELATIVISTIC SPHERICAL FUNCTIONS

I. S. SHAPIRO
Institute of Theoretical and Experimental Physics
of the USSR Academy of Sciences, Moscow, USSR

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The importance of the investigation of the analytical properties of the scattering amplitude as function of the angular momentum [1] has become clear off late. For high energies the expansion of the amplitude in three-dimensional spherical functions can be used effectively in those cases when the momentum transfer in the scattering channel and consequently the energy in the annihilation channel is small [2-4]. If the momentum transfer is comparable with energy the study of the asymptotic behaviour of amplitudes with the aid of the above expansion becomes rather difficult. This raises the question of the expansion of the amplitude in the proper functions of the four-dimensional angular momentum or, to be more specific, in the irreducible representations of the Lorentz homogeneous group.

For simplicity we consider here the scattering amplitude of zero spin particles with equal masses x. The generalization to the case of particles with spins and different masses does not involve any difficulties.

From the existence of the dispersion relations with respect to the momentum transfer for the scattering amplitude $U(t, s)$ (t, s being the usual Mandelstam variables) follows the convergence (at any fixed S) of the invariant integral

$$N = \int \left| \frac{U(t,s)}{(t-a)^n} \right|^2 \frac{d^3p}{\epsilon} < \infty , \quad a > 0 , \quad n \geq 0 , \tag{1}$$

where $\vec{p}$ and ϵ are the momentum and energy of the scattered particle. If we introduce, having designated by p^0 and p the four momenta of the incident and scattered particles, the "cosine" Z

$$Z = \frac{p^0 p}{x^2} = 1 - \frac{t}{2x^2} , \tag{2}$$

the integral (1) can be rewritten in the form

$$N = 4\pi x^2 \int_1^\infty |f(Z,s)|^2 \sqrt{Z^2 - 1}\, dZ , \tag{1a}$$

where

$$f(Z,s) = \frac{U(t,s)}{(t-a)^n} . \tag{3}$$

It should be noted that since $d^3p/x^2\epsilon$ is the unit sphere surface element, then, regarding p^0 as fixed, we can consider eq. (1) as an integral over the surface of the four-dimensional sphere of the function of the 4-vector p. Thus our problem is quite similar to finding the expansion of the scattering amplitude in three-dimensional spherical functions. It is known that in the latter case the