1. The Hamiltonian for a classical harmonic oscillator can be written in many different forms, such as (use $\omega = \sqrt{k/m}$)

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$H = \frac{1}{2}(P^2 + Q^2)\hbar\omega$$

a. Find a canonical transformation from the familiar coordinate $x$ and momentum $p$ to their dimensionless counterparts $Q, P$.

b. Check that the Hamilton equations of motion are satisfied for both sets of coordinates.

c. To go from classical to quantum mechanics, Schrödinger tells us to replace $x \rightarrow \hat{x}$ and $p \rightarrow \hat{p}$ and require that these operators satisfy the commutation relations $[\hat{p}, \hat{x}] = \hbar/i$. What commutation relations does this force the operators $\hat{Q}, \hat{P}$ to have?

2. The Lorentz force on a particle of charge $q$ moving with velocity $\mathbf{v}$ in the presence of electric and magnetic fields $\mathbf{E}, \mathbf{B}$ is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

a. Express $\mathbf{E}$ and $\mathbf{B}$ in terms of the vector and scalar potential $\mathbf{A}$ and $\phi$.

b. Find an expression for the total derivative $d\mathbf{A}/dt$ in terms of its partial derivatives and $\mathbf{v}$. Explain exactly how this expression is derived.

c. Expand $\frac{1}{c}\mathbf{v} \times (\nabla \times \mathbf{A})$.

d. Plug the expressions from b. and c. into a. to find an expression of the form $\mathbf{F} = -\nabla U +$ the total time derivative of something.
e. What is U? What is “something” and why isn’t it important?

f. Construct the Lagrangian for the motion of a charged particle in the presence of an electric and magnetic field: \( L = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v} - U \).

g. Construct the momentum conjugate to \( x \) in the usual way: \( \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{v}}} \).

h. Use the standard (Legendre) transformation to construct the Hamiltonian from \( L \): \( H = \mathbf{p} \cdot \dot{\mathbf{v}} - L \), and express \( H \) as a function of \( x, \mathbf{p}, \mathbf{A} \).
i. Verify that the Hamiltonian equations of motion give the correct result.

3. Compute the thermal expectation value for a quantum harmonic oscillator with angular frequency \( \omega \). Recall that

i. The occupation probability of a state with energy \( E_n \) is proportional to the Boltzmann fact: \( P_n \simeq e^{-\beta E_n} \).

ii. Show \( P_n = e^{-\beta E_n}/Z \), where \( Z \) is the partition function.

iii. Use \( E_n = (n + \frac{1}{2})h\omega \).

a. Show \( \langle E \rangle = (\langle n \rangle + \frac{1}{2})h\omega \). What is \( \langle n \rangle \)?

b. How many modes of the electromagnetic field exist in the (angular) frequency range \( \omega \) to \( \omega + d\omega \)?

c. Compute the mean energy per unit volume in the electromagnetic field when it is in thermal equilibrium at temperature \( T \) \( (\beta = 1/kT) \) with its surroundings. Show that this energy consists of two terms, one of which is proportional to \( T^4 \) and the other of which diverges.

4. The nonrelativistic hamiltonian that describes the interaction of a charged particle with the electromagnetic field is

\[
H = \frac{1}{2m}(\mathbf{p} - \frac{q}{c}\mathbf{A})^2 + q\phi
\]

In quantum mechanics \( \mathbf{p} \rightarrow \langle h/i \rangle \nabla \) and the hamiltonian acts on a wavefunction \( \psi(x,t) \).

a. Assume that the wavefunction is changed by a constant phase \( \psi \rightarrow \psi' = e^{i\alpha}\psi \). Show that \( \psi' \) satisfies the original Schrödinger equation with the original vector potential \( \mathbf{A} \).

b. Assume that the wavefunction is changed by a nonconstant phase \( \psi \rightarrow \psi' = e^{i\alpha(x,t)}\psi \). Show that \( \psi' \) does not satisfy the original Schrödinger equation with the original vector potential \( \mathbf{A} \).

c. Show that \( \psi' \) does satisfy the original Schrödinger equation, but with a new vector potential \( \mathbf{A}' = \mathbf{A} + \nabla \chi \). How is the gauge term \( \chi(x,t) \) related to the phase term \( \alpha(x,t) \)?

d. How does the scalar potential change: \( \phi \rightarrow \phi' + \text{?} \)?