QUANTUM MECHANICS I

PHYS 516

Solution to Midterm Exam, Feb. 12, 2016

1. Scaling: The energy E and Bohr radius a_B for a hydrogen atom in its ground state are $E = -me^4/2\hbar^2 = -13.6\ eV$ and $a_B = \hbar^2/me^2 = 0.529$ Å. Estimate the ground state energy and radius of positronium.

Solution: The mass in the expressions above is the electron-proton reduced mass, essentially m_e. In positronium the reduced mass is $(m_e m_{pos.}/(m_e + m_{pos.}) = \frac{1}{2}m_e$. This means the binding energy is reduced by half and the size is increased by a factor of 2: $E = -6.8$ eV and $a = 1.06$Å.

2. Linear Chain: In one dimension, n particles, each of mass m, are coupled to each other by springs of spring constant k. The end masses are connected to brick walls by springs with spring constant k.

a. Guess the nature of the normal modes.

b. Construct the dispersion relation $\omega(\phi)$, where ϕ is an appropriate mode index $\phi = i2\pi m/(n+1)$.

c. Quantize this normal mode problem.

Solution: If the displacement of the ith particle from its equilibrium position is x_i, with $1 \leq i \leq n$, then in the mth normal mode ($1 \leq m \leq n$), $x_i \simeq \sin\left(\frac{im\pi}{n+1}\right)$.

The Euler-Lagrange equation for x_i is

$$-kx_{i-1} + (2k - m\omega^2)x_i - kx_{i+1} = 0$$

Using standard trigonomtric identities we find

$$2k - 2k\cos\left(\frac{m\pi}{n+1}\right) - m\omega^2 = 0$$

The result is $\omega^2 = 4(k/m)\sin^2\left(\frac{m\pi/2}{n+1}\right)$

The quantization step is

$$\mathcal{H} = \sum_{m=1}^{n} (a_m^\dagger a_m + \frac{1}{2})\hbar\omega(m)$$

with $\omega(m)$ as constructed from the equation above.

3. More Harmonic Oscillators: Three harmonic oscillators have energy spacing $\hbar\omega_1 = \hbar\omega_2 = 400MeV$ and $\hbar\omega_3 = 600MeV$. These oscillators share three excitations ($n_1 + n_2 + n_3 = 3$). Draw an energy level diagram, clearly indicating the energies and the degeneracies.

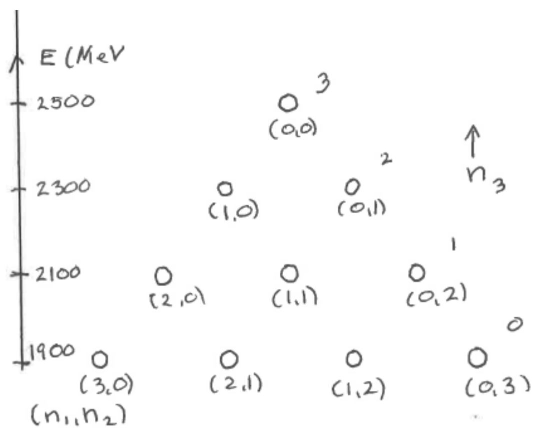

Figure 1: Energy level spectrum of a 3-dimensional harmonic oscillator with $N = n_1 + n_2 + n_3 = 3$ escitations with energies $\hbar\omega_1 = \hbar\omega_2 = 400$ MeV and $\hbar\omega_3 = 600$ MeV.

4. Diatomic Molecules: An imaginary diatomic molecule has an energy level spectrum given by the analytical expression

$$E(n,l) = \frac{(n+\frac{1}{2})\hbar\omega}{1+\alpha(n+\frac{1}{2})} \times \frac{l(l+1)\hbar^2}{2I_0(1-\beta l(l+1))}$$

Here I_0 is the moment of inertia and α, β are dimensionless.

Write down the 22 component $D_{2,2}$ in the Dunham energy expansion $E(n,l) = \sum_{p,q} D_{p,q}(n+\frac{1}{2})^p \,[l(l+1)]^q$.

Solution: Expanding the first term gives

$$\frac{(n+\frac{1}{2})\hbar\omega}{1+\alpha(n+\frac{1}{2})} = (n+\frac{1}{2})\hbar\omega - (n+\frac{1}{2})\hbar\omega \times \alpha(n+\frac{1}{2}) + \cdots$$

Expanding the second term gives

$$\frac{l(l+1)\hbar^2}{2I_0(1-\beta l(l+1))} = \frac{l(l+1)\hbar^2}{2I_0}\left(1+\beta l(l+1)+\cdots\right)$$

Multiply the two expressions and pick out the coefficient of $(n+\frac{1}{2})^2\left[l(l+1)\right]^2$ to find

$$D_{2,2} = -\hbar\omega \times \frac{\hbar^2}{2I_0} \times \alpha \times \beta$$