QUANTUM MECHANICS I

PHYS 516

Problem Set # 6

Distributed: March 12, 2012

Due: Noon, March 23, 2012

In the presence of a magnetic field $\mathbf{B}$ the kinetic energy for a particle of mass m and charge q is obtained by the simple transformation (Principal of Minimal Electromagnetic Coupling)

$$p \to \Pi = p - \frac{q}{c}\mathbf{A} : \qquad\qquad \frac{p^2}{2M} \to \frac{\Pi^2}{2M}$$

Here $\mathbf{A}$ is the *vector potential*, not uniquely defined by $\mathbf{B} = \nabla \times \mathbf{A}$.

A particle of mass M and charge q is confined to move in an annulus in the plane (c.f., Ballentine, pp. 323-325). The inner radius is a and the outer radius is b. A cylindrically symmetric magnetic field threads the hole in the annulus. The total magnetic flux through the hole is Φ. The vector potential in the plane that describes the magnetic field is

$$\mathbf{A} = \frac{\Phi}{2\pi r^2}\ \mathbf{k} \times \mathbf{r} = \frac{\Phi}{2\pi}\ \frac{(-y, x, 0)}{x^2 + y^2}$$

a. Show that $\nabla \times \mathbf{A} = \mathbf{0}$.
b. Show that $\nabla \cdot \mathbf{A} = \mathbf{0}$.
c. Write down Schrödinger's equation.
d. Transform to cylindrical coordinates.
e. Make the following ansatz:

$$\psi(r, \theta) = \frac{1}{\sqrt{r}} f(r) e^{im\theta}$$

Argue that m must be an integer for the wavefunction to be single-valued.

f. Show that the radial wave equation reduces to the form

$$\left\{ \frac{d^2}{dr^2} - \frac{K}{r^2} \left(m - \frac{\Phi}{\Phi_0} \right)^2 + \frac{2ME}{\hbar^2} \right\} f(r) = 0$$

where $\Phi_0 = 2\pi\hbar c/q = hc/q$ is the natural unit of magnetic flux. What is K?

g. Show that the radial wave equation remains unchanged under the transformation $\Phi \to \Phi + \Phi_0$ and $m \to m + 1$.

h. Compute the five lowest energy radial wavefunctions $f(r)$ for $(m - (\Phi/\Phi_0)) = 0.0, 0.5, 1.0, 1.5, 2.0$. You can either use Bessel functions (not recommended) or diagonalize a basis set of sine functions that vanish at the edges (i.e., $\sin\left[n\pi(r-a)/(b-a)\right]$). Use $M = \hbar = 1$, $a = 5, b = 10$.

i. Compute the azimuthal current density using

$$\mathbf{j} = \frac{\hbar}{2Mi} \left[\psi^* \nabla \psi - \psi \nabla \psi^* \right] - \frac{q}{Mc} \mathbf{A} \psi^* \psi$$

and show that it is proportional to $\frac{\hbar}{Mr}(m - (\Phi/\Phi_0))$. What is the proportionality constant?

k. Evaluate $\oint \mathbf{j} \cdot d\mathbf{r}$ around a closed circular loop of radius d centered on the symmetry axis.