THE STORY OF SPIN

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LECTURE SIX

Pauli-Weisskopf and the Yukawa Particle

There Is No Reason for Nature to Reject Particles with Spin Zero

With this lecture I return to the main stream which was previously discussed. In the third lecture I described "why nature is not satisfied by a simple point charge." Dirac answered this question. Maybe I am too gossipy, but I suspect that when Pauli saw Dirac's work, he thought he had been scooped by him. Pauli himself was trying to incorporate spin into quantum mechanics and introduced the Pauli matrices, but he could not arrive at the complete theory. He also clearly recognized the necessity of making the theory relativistic and yet could not achieve it. He was known to be a perfectionist who severely criticized other people's work, but he had no alternative other than to publish his unsatisfactory ad hoc theory. Now on the other hand, Dirac established his theory with totally unexpected acrobatics and solved all the problems on spin as well as making the theory relativistic. I do not believe Pauli could have been so ascetic as to have accepted this graciously.

However, in 1934 Pauli ended up retaliating against Dirac. Pauli showed that Dirac's argument that nature is satisfied only by the Dirac equation and not by the Klein-Gordon equation is incorrect. According to him, the Klein-Gordon equation is not in contradiction with the framework of quantum mechanics, and there is no reason nature abhors particles with spin 0.

I would like to talk today on how this came about, but we need quite a bit of preparation. The first item is the quantization of the wave field, and the other is the problem of Dirac's negative energy.

Probably you have had the same experience, but the "wave," when we study wave mechanics or the wave function ψ, is discussed sometimes as if it were a real wave in the three-dimensional space in which we live and sometimes as if it were an abstract wave in the configuration space. Have you ever worried as to which is the case? Such confusion with the concept occurred quite a bit

Figure 6.1 Wolfgang Pauli, 1940 passport photo. [Courtesy of AIP Emilio Segrè Visual Archives]

at the beginning of the development of quantum mechanics. Indeed, if we look at Schrödinger's series of papers "Quantization as an Eigenvalue Problem," we find that Schrödinger himself was going back and forth between these two ideas. However, by and large he tended to think that his ψ was a wave in three-dimensional space. For example, he considered $e\psi^*(\mathbf{x})\psi(\mathbf{x})$ as the charge density which actually exists in space and tried to treat the bulk of the density as an electron. This idea, however, did not work because $\psi^*\psi$ will spread with time and the density becomes diffuse.

On the other hand, when quantum mechanics was completed in the form of transformation theory, Schrödinger's ψ was established as a representation of the state vector of the mechanical system. It was also called the probability amplitude, which is a function of the generalized coordinates describing the mechanical system, and therefore the wave expressed by ψ exists in the abstract coordinate space and not in our three-dimensional space. Therefore, for two particles, ψ is a function $\psi(\mathbf{x}_1, \mathbf{x}_2)$ of the coordinate $\mathbf{x}_1$ of the first particle and the coordinate $\mathbf{x}_2$ of the second particle, and this is a wave in a six-dimensional space. Furthermore, more generally, the variables which constitute the argument of ψ are not limited to the spatial coordinates $\mathbf{x}_1, \mathbf{x}_2, \ldots$ but could be the generalized coordinates of Lagrange or even the more abstract canonical coordinates of Hamilton's mechanics.

Therefore, it looks as if Schrödinger's idea of a wave of real matter seems to have been completely superseded. The idea that the radiation wave does indeed exist in our three-dimensional space but the matter wave does not was about to become orthodox.

However, this situation suddenly changed when the idea of quantization of the wave field was introduced. Namely, it was established that the concept of a matter wave actually existing in space is just as valid as that of a light wave existing in space.

Quantum mechanics was initially developed to study material particles such as electrons and nuclei. And the phenomena of atoms emitting, absorbing, and scattering radiation were treated through the analogy with classical mechanics with the assumption that the matrix elements of the electronic coordinates correspond to the Fourier components of the classical electron motion; there was no other recourse than to treat the whole problem by analogy to classical theory. However, when quantum mechanics was completed in the form of transformation theory, it was ready to be applied to the radiation field. There was an attempt to consider a mechanical system that includes atoms as well as a radiation field and to apply quantum mechanics also to the field, treating the interaction of radiation and matter consistently. It was Dirac who initiated this attempt in 1927.

The idea of treating the radiation field quantum mechanically was by no means new. Already at the beginning of quantum theory was Debye's idea that if you decompose the radiation field into plane waves and make its amplitude discrete such that it satisfies Planck's condition

$$E_\nu = N_\nu h\nu, \qquad N_\nu = 0, 1, 2, \ldots,$$

then you can derive Planck's formula. Therefore, instead of simply using Planck's condition ad hoc, we can reconsider the amplitude of the wave as a matrix (or Dirac's q-number) and apply the new quantum mechanics. Then the condition $E_\nu = N_\nu h\nu$ will be naturally derived, and we can interpret N_ν as the number of photons with energy $h\nu$. If we do this, the particle nature of radiation will automatically emerge. Indeed soon after Heisenberg devised matrix mechanics, Born, Jordan, and Heisenberg, who completed matrix mechanics, themselves proposed to consider the electric and magnetic fields **E** and **H** also as matrices (although at that time the idea of probability amplitude did not yet exist, and they could not apply their idea to the emission, absorption, and scattering of radiation).

Needless to say, Dirac naturally started from the idea of Debye and Born. This initial step could have been taken by any of us mortals (regardless of whether we can complete it or not), but Dirac introduced one further idea which is characteristic of him. It is the weird idea which later came to be known as *second quantization*. I am afraid that I must digress from spin, but this weird idea is very typical of Dirac, and with this idea the *matter wave* is incorporated into the three-dimensional space, so let me spend some time explaining it.

Dirac first considered the Schrödinger equation for one particle. As you know, it is

$$\left[H(\mathbf{x},\mathbf{p}) + \frac{\hbar}{i}\frac{\partial}{\partial t}\right]\psi(\mathbf{x},t) = 0 \tag{6-1}$$

$$H(\mathbf{x},\mathbf{p}) = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{x}), \qquad \mathbf{p} = \frac{\hbar}{i}\nabla.$$

According to his transformation theory, $\psi(\mathbf{x},t)$ represents the state vector of the system at time t. Now we consider some mechanical quantity (in Dirac's nomenclature, an observable) $G(\mathbf{x},\mathbf{p})$, and let us call its eigenvalue g_n and eigenfunction $\phi_n(\mathbf{x})$. Here, $n = 1, 2, 3, \ldots$ Since the $\phi_n(\mathbf{x})$ constitute a complete, orthogonal system, we can expand $\psi(\mathbf{x},t)$ as

$$\psi(\mathbf{x},t) = \sum_n a_n(t)\phi_n(\mathbf{x}). \tag{6-2}$$

Then if we measure the observable $G(\mathbf{x},\mathbf{p})$ at time t, the probability of obtaining the value g_n is

$$P_n(t) = \left|a_n(t)\right|^2. \tag{6-3}$$

This was a conclusion from the transformation theory. Here ψ and ϕ_n are all normalized to 1. $H(\mathbf{x},\mathbf{p})$ represents the energy of the system, and in terms of the matrix elements of H,

$$H_{n,n'} = \int \phi_n^*(\mathbf{x})H(\mathbf{x},\mathbf{p})\phi_{n'}(\mathbf{x})dv, \tag{6-4}$$

the expectation value of H is given by

$$\begin{aligned}\langle H\rangle &\equiv \int \psi^*(\mathbf{x},t)H(\mathbf{x},\mathbf{p})\psi(\mathbf{x},t)dv \\ &= \sum_{n,n'} a_n^* H_{n,n'}a_{n'}.\end{aligned} \tag{6-5}$$

Here it is easy to see that $\langle H\rangle$ is independent of time.

Using the matrix element (6-4), we can derive the time dependence of $a_n(t)$ as

$$\frac{da_n(t)}{dt} = \frac{1}{i\hbar}\sum_{n'} H_{n,n'}a_{n'}(t) \tag{6-6}$$

from (6-1). Its complex conjugate becomes

$$\frac{da_n^*(t)}{dt} = -\frac{1}{i\hbar}\sum_{n'} a_{n'}^*(t) H_{n',n}. \tag{6-6)*}$$

If we examine (6-6) and (6-6)*, we see that we can rewrite them using $\langle H\rangle$ of (6-5), namely

$$\frac{da_n}{dt} = \frac{1}{i\hbar}\frac{\partial\langle H\rangle}{\partial a_n^*}, \qquad \frac{da_n^*}{dt} = -\frac{1}{i\hbar}\frac{\partial\langle H\rangle}{\partial a_n}. \tag{6-7}$$

From these, if we treat a_n like the coordinate variable and

$$\pi_n = i\hbar a_n^* \tag{6-8}$$

like its conjugate momentum, and consider

$$\langle H\rangle = \frac{1}{i\hbar}\sum_{n,n'} \pi_n H_{n,n'} a_n' \tag{6-9}$$

as the Hamiltonian, then we can show that a_n and π_n satisfy the canonical equations of motion

$$\frac{da_n}{dt} = \frac{\partial\langle H\rangle}{\partial \pi_n}, \qquad \frac{d\pi_n}{dt} = -\frac{\partial\langle H\rangle}{\partial a_n}. \tag{6-10}$$

So far we have considered one mechanical system consisting of one particle, but let us follow Dirac here and consider an ensemble composed of N mechanical systems each containing one particle. If we do so, the expectation value of the number of systems in the ensemble which have the value g_n for the observable G at one time is given as NP_n, namely

$$N_n \equiv NP_n = N|a_n|^2. \tag{6-3')}$$

Therefore, if we set

$$A_n = N^{1/2}a_n \quad \text{and} \quad A_n^* = N^{1/2}a_n^*, \tag{6-11}$$

then we get

$$N_n = A_n^* A_n, \tag{6-3'')}$$

and if we set, corresponding to (6-8),

$$\Pi_n = i\hbar A_n^*, \tag{6-8'}$$

and, corresponding to (6-9),

$$\overline{H} = \frac{1}{i\hbar} \sum_{n,n'} \Pi_n H_{n,n'} A_{n'}, \tag{6-9'}$$

then we obtain, corresponding to (6-10),

$$\frac{dA_n}{dt} = \frac{\partial \overline{H}}{\partial \Pi_n}, \qquad \frac{d\Pi_n}{dt} = -\frac{\partial \overline{H}}{\partial A_n}. \tag{6-10'}$$

If we examine (6-10'), here also we can regard A_n and Π_n as canonical variables, which satisfy canonical equations with $\overline{H}$ as the Hamiltonian. We have the relation

$$\sum_n |a_n|^2 = \int \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)dv = 1 \tag{6-12}$$

for a_n, while for A_n,

$$\sum_n |A_n|^2 = N. \tag{6-12'}$$

Dirac says that if you worry about the fact that A_n and Π_n are complex numbers, then use N_n and Θ_n defined by

$$A_n = N_n^{1/2} e^{i\Theta_n/\hbar} \qquad A_n^* = N_n^{1/2} e^{-i\Theta_n/\hbar}. \tag{6-13}$$

Dirac proved that these N_n and Θ_n are canonical variables conjugate to each other. When we use these variables, the Hamiltonian is

$$\overline{H} = \sum_{n,n'} N_n^{1/2} e^{-i\Theta_n/\hbar} H_{n,n'} N_{n'}^{1/2} e^{i\Theta_{n'}/\hbar}. \tag{6-9''}$$

Here Dirac performed his characteristic acrobatics. Namely, he redefined these A_n and Π_n as quantum mechanical q-numbers rather than as ordinary numbers. That is, he introduced the canonical commutation relation between A_n and Π_n to quantize the problem,

$$A_n \Pi_{n'} - \Pi_{n'} A_n = i\hbar \delta_{nn'}, \tag{6-14}$$

$$A_n A_{n'} - A_{n'} A_n = \Pi_n \Pi_{n'} - \Pi_{n'} \Pi_n = 0.$$

Why do I call this acrobatics? After all, the Schrödinger equation (6-1) is already the result of quantization. Therefore, all of equations (6-10) and (6-10') derived from that are already quantized. Why must you quantize it once more as the name second quantization suggests? We mortals stand bewildered here. However, there is no use being bewildered, so let us try to discover why we feel bewildered. We shall find the following.

The quantities which are expressed by q-numbers in quantum mechanics, be they the coordinate q, the momentum p, the energy H, or the quantity G that we just discussed, are all observables. The concept of observable was introduced by Dirac, and these are the quantities which are directly measurable by some experiments. However, quantities like Π_n, defined in (6-3), are nothing like that. In order to determine them, we repeat the measurement of observable G many times, accumulate a body of data, and then examine which fraction of measurements give g_n. Therefore Π_n is not a quantity which is directly measurable by experiment, and therefore it is not an observable. In that sense, none of a_n, a_n^*, π_n are observables.

We can say the same thing about N_n, A_n, A_n^*, and Π_n. First of all, when I said "an ensemble composed of N mechanical systems each with one particle," I meant by *ensemble* an imagined ensemble, and there is no need to have a real mechanical system composed of N particles in front of you. We can consider one mechanical system and repeat the measurement of G many times under the same conditions and then obtain the number of times N_n in which the value of G was measured to be g_n. Or we can make the mechanical system number one on the first day and measure G and destroy the system after the measurement, make a second mechanical system on the second day and measure the value of G and destroy that system, make a third mechanical system on the third day, and so on. Therefore, the ensemble we are considering here is a so-called virtual ensemble in statistics. When I say number of measurements N_n, it is not a quantity which itself is obtained by measuring an observable, but it is a number related to the accumulation of data on many repeated measurements of observables.

The reason why we were bewildered by Dirac's second quantization, which regards N_n, A_n, and Π_n as q-numbers, lies right here. Some of you might accept the second quantization without any difficulty. Such a person has to be either as great as Dirac or a happy-go-lucky type who, although he does not consider anything deeply, feels as if he understands everything.

Then on what basis did Dirac dare to quantize N_n, A_n, and Π_n in spite of these problems? I think the answer is as follows.

We know the fact that the statistical conclusion for virtual ensembles often agrees with that for real ensembles. Therefore, we may well expect that in the present problem there may also be such agreement. If so, we can apply our conclusion for a "virtual ensemble composed of N mechanical systems each with one particle" to the "mechanical system of N particles which are not mutually interacting." Now in this real ensemble we can consider N_n as an observable because the determination of N_n for a mechanical system composed of N particles amounts to the measurement of G for particle 1, 2, 3, ..., N, i.e., $G(\mathbf{x}_1, \mathbf{p}_1)$, $G(\mathbf{x}_2, \mathbf{p}_2)$, $G(\mathbf{x}_3, \mathbf{p}_3), \ldots, G(\mathbf{x}_N, \mathbf{p}_N)$. Here the G of the N particles all commute, and therefore according to the transformation theory, we can measure all N of the G's simultaneously and regard it as one set of measurements. Furthermore, observable N_n is not an analytic function but is a function of N observables G (since the value of N_n is determined when the N G values which mutually commute are measured, N_n is a function of all the N G's), and therefore N_n can be determined by one set of measurements. For this reason we can consider N_n as a q-number in our real ensemble and likewise A_n and Π_n.

For these reasons, what Dirac did was a heuristic study to find the answer to the question whether it is possible to describe a many-particle system using the q-numbers N_n and their conjugates Θ_n or the q-numbers A_n and Π_n, and for that purpose to use equation (6-10') as the fundamental equation starting from the Hamiltonian (6-9'). Namely, he is using the second quantization as a heuristic means to see if such an approach is possible and only in this context.

Since the second quantization is a heuristic logic, we must examine whether the theory using the Hamiltonian (6-9') or (6-9") indeed agrees with the conclusion obtained from the treatment of many-particle systems in the ordinary way. [In the process of quantizing (6-9"), the order of $N_n^{1/2}$ and $e^{\pm i\Theta_n/\hbar}$ must be considered, but the right order will be given later in (6-17').] Here by ordinary way I mean the method of considering ψ in coordinate space which for the N particles gives the Schrödinger equation

$$\left[H(\mathbf{x}_1, \mathbf{p}_1) + H(\mathbf{x}_2, \mathbf{p}_2) + \ldots + H(\mathbf{x}_N, \mathbf{p}_N) + \frac{\hbar}{i}\frac{\partial}{\partial t}\right] \cdot \psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = 0. \tag{6-15}$$

Dirac actually gives in his paper a proof that this agreement does result if he takes only the solution $\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ which is symmetric with respect to the interchange of particles. By this procedure it has been demonstrated that the solution of the problem by using Hamiltonian (6-9') and commutation relation (6-14) is equivalent to solving (6-15) for an N-boson system with the Hamiltonian

$$H = \sum_{\nu=1}^{N} H(\mathbf{x}_\nu, \mathbf{p}_\nu). \tag{6-16}$$

Here the probability amplitude that appears when we use Hamiltonian (6-9') or (6-9") is a function of the type of coordinate used, namely A_n or N_n

$$\psi(A_1, A_2, \ldots, A_n, \ldots) \quad \text{or} \quad \psi(N_1, N_2, \ldots, N_n, \ldots). \tag{6-17}$$

In particular, the latter is more commonly used, and the Schrödinger equation for that case is

$$\left[\sum_{n,n'} N_n^{1/2} e^{-i\Theta_n/\hbar} H_{n,n'} e^{i\Theta_{n'}/\hbar} N_{n'}^{1/2} + \frac{\hbar}{i}\frac{\partial}{\partial t}\right] \cdot \psi(N_1, N_2, \ldots, N_n, \ldots) = 0. \tag{6-17'}$$

Dirac showed that the operator $e^{\pm i\Theta/\hbar}$ in this equation has the property

$$e^{\pm i\Theta/\hbar}\psi(N) = \psi(N \pm 1).$$

Actually this proof was not rigorous, but it is very Dirac-like and very interesting as heuristic theory, so let me introduce it here.

Proof: Since Θ is the momentum conjugate to N, we can consider it to be $\hbar\partial/i\partial N$ when it is applied to a function of N. Then we have

$$e^{\pm i\Theta/\hbar} = e^{\pm\partial/\partial N} = 1 \pm \frac{\partial}{\partial N} + \frac{1}{2!}\frac{\partial^2}{\partial N^2} \pm \frac{1}{3!}\frac{\partial^3}{\partial N^3} + \ldots.$$

By Taylor's theorem

$$\psi(N) \pm \psi'(N) + \frac{1}{2!}\psi''(N) \pm \frac{1}{3!}\psi'''(N) + \ldots = \psi(N \pm 1),$$

therefore

$$e^{\pm i\Theta/\hbar}\psi(N) = \psi(N \pm 1). \qquad \text{Q.E.D.}$$

Since the newly discovered theory has now been proved to be correct for bosons, the use of heuristic theory notwithstanding, there is no reason for us to hesitate to use it. We can proceed with it with confidence. Therefore we shall transform the Hamiltonian (6-9'), the equations of motion (6-10'), and the commutation relation (6-14) together with the relation between A_n and N (6-12') so we can use them for further discussions. Remembering (6-2), we define the "wave function" $\Psi(\mathbf{x})$ and its "conjugate momentum function" $\Pi(\mathbf{x})$, which are q-numbers, by

$$\Psi(\mathbf{x}) = \sum_n A_n\phi_n(\mathbf{x}) \tag{6-2'}$$

$$\Pi(\mathbf{x}) = \sum_n \Pi_n\phi_n^*(\mathbf{x}).$$

Then we obtain from (6-14) the canonical commutation relations between $\Psi(\mathbf{x})$ and $\Pi(\mathbf{x})$

$$\begin{aligned}\Psi(\mathbf{x})\Pi(\mathbf{x}') - \Pi(\mathbf{x}')\Psi(\mathbf{x}) &= i\hbar\, \delta(\mathbf{x}-\mathbf{x}')\\ \Psi(\mathbf{x})\Psi(\mathbf{x}') - \Psi(\mathbf{x}')\Psi(\mathbf{x}) &= 0\\ \Pi(\mathbf{x})\Pi(\mathbf{x}') - \Pi(\mathbf{x}')\Pi(\mathbf{x}) &= 0,\end{aligned} \tag{6-14'}$$

and we can derive the equation of motion for Ψ as

$$\left[H(\mathbf{x},\mathbf{p}) + \frac{\hbar}{i}\frac{\partial}{\partial t}\right]\Psi(\mathbf{x},t) = 0. \tag{6-1'}$$

Furthermore, since we see

$$\Pi(\mathbf{x}) = i\hbar\Psi^{\dagger}(\mathbf{x}), \tag{6-8''}$$

the equation of motion of $\Pi(\mathbf{x})$ is nothing but the complex conjugate of (6-1'). (Since Ψ is a q-number, we use † instead of * to express the complex conjugate.) Furthermore, the Hamiltonian (6-9') becomes

$$\overline{H} = \frac{1}{i\hbar}\int \Pi(\mathbf{x})H(\mathbf{x},\mathbf{p})\Psi(\mathbf{x})dv = \int \Psi^{\dagger}(\mathbf{x})H(\mathbf{x},\mathbf{p})\Psi(\mathbf{x})dv, \tag{6-5'}$$

and the relation (6-12') becomes

$$N = \int \Psi^{\dagger}(\mathbf{x},t)\Psi(\mathbf{x},t)dv. \tag{6-12''}$$

This N commutes with $\overline{H}$, and therefore we see it is independent of time.

Now if we look at (6-1') just obtained, we see it has a similar form to that of the probability amplitude of one particle, and if we look at the Hamiltonian (6-5'), then we see it has the same form as equation (6-5) for the expectation value of the energy for one particle. We should not forget, however, that although they are similar in form, their meanings are entirely different. Namely, (6-1') and (6-5') are related to the observables of "a mechanical system composed of N bosons," and both are relations between q-numbers, whereas (6-1) and (6-5) are related to the probability amplitude and the expectation value of one particle. We should note here that neither (6-1'), (6-5'), nor the commutation relation (6-14') contains the boson number N at all. We therefore can consider those relations as the fundamental formulas for "the mechanical system with an arbitrary number of bosons." Because of this, the wave function $\Psi(\mathbf{x})$, unlike the wave ψ in the coordinate space, is always a function of $\mathbf{x}$ in three-dimensional space, which is independent of the number of particles. We therefore can regard the wave function Ψ, which is our q-number, as the wave which actually exists in the three-dimensional space in which we live. Where does the number of particles appear then? The answer is in (6-12"). Namely, the number of particles appears related to the amplitude of Ψ.

We now know that the wave field which has the Hamiltonian (6-5') and is quantized by the commutation relation (6-14') is equivalent to the system of bosons with the Hamiltonian (6-16). Indeed, if we consider Ψ as a q-number and define A_n by (6-2'), then the eigenvalues of the q-number N_n, defined by

$$N_n = A_n^{\dagger} A_n, \tag{6-18}$$

are

$$\text{eigenvalues of } \ N_n = 0, 1, 2, \ldots, \tag{6-18'}$$

and therefore we see clearly the particle nature of the field. Furthermore, if we adopt Schrödinger's idea that $e\psi^*(\mathbf{x})\psi(\mathbf{x})$ is the charge density which actually exists in space and define the observable

$$\rho(\mathbf{x}) = e\Psi(\mathbf{x})^{\dagger}\Psi(\mathbf{x}), \tag{6-19}$$

(so far we have been using as electric charge $-e$, but today I shall use e for the charge of a particle regardless of the sign) then we see that the eigenvalues of its integral over arbitrary volume V

$$\rho_V = \int_V \rho(\mathbf{x}) d\mathbf{x} \tag{6-19'}$$

are 0, $1e$, $2e$, $3e$, . . . This shows that our mechanical system has the property of an assembly of particles with electric charge e. We should note here that corresponding to the spreading of Schrödinger's $e\Psi^*(\mathbf{x})\Psi(\mathbf{x})$ in time, the expectation value of $e\Psi^{\dagger}(\mathbf{x})\Psi(\mathbf{x})$ is also blurred with time, and therefore the expectation value of ρ_V also may take a nonintegral value and gradually approaches 0. However, the eigenvalue of ρ_V will never take values other than 0 or positive integers. Therefore, although the expectation values may blur, the particle nature of electric charge is always conserved.

In this way Schrödinger's unfulfilled wish, namely not to place the wave $\psi(\mathbf{x})$ in the configuration space but to welcome it to the three-dimensional space, has now been realized by using the quantized $\Psi(\mathbf{x})$ rather than $\psi(\mathbf{x})$. Thus guided by his heuristic method, Dirac discovered that the field equation satisfied by

$\Psi(\mathbf{x})$ has the same form as the equation satisfied by $\psi(\mathbf{x})$. You should never forget, however, that even though the form of the equations is the same, ψ is the probability amplitude of one particle and is therefore a *c*-number, whereas Ψ, which describes the wave field, is a *q*-number, and they are conceptually entirely different things. Furthermore, as I shall tell you later, the coincidence of the form of the equations is limited only to cases in which the interactions between particles are neglected; if there are interactions, ψ and Ψ are not only conceptually different, but the equations satisfied by them have mathematical properties which are essentially different from each other. It is often said, "by the second quantization of ψ we obtain Ψ," but this statement is wrong for this reason. In my opinion, just as there are Maxwell equations which are not quantized, it is better to consider that there exist some equations from the beginning satisfied by nonquantized Ψ, and those equations agree with the equations of ψ only when there is no interaction.

Anyhow, it was a great discovery to have found that the mechanical system with the Hamiltonian (6-5'), i.e., the system of the wave field with equation (6-1'), and the mechanical system with Hamiltonian (6-16), i.e., a particle system composed of N particles, give exactly the same answer if we quantize the former by (6-14') and adopt only the symmetric wave function for the latter. They are entirely equivalent. It is great because from this we can establish a Nishida-like theme[1] in quantum mechanics that says a wave is a particle and a particle is a wave without any contradiction and obtain its complete mathematical expression.

Dirac's heuristic study in the form I have considered so far does not work for a real ensemble in which there is an interaction between particles. This is because in the virtual ensemble, which he used as a starting point, particle interactions do not play any role. After all, an individual mechanical system in his virtual ensemble is a system of *one* particle. Therefore, there is no interaction in the system. Also, the ensemble is virtual, and we may consider, for example, that the first system exists on the first day only, the second on the second day only, the third on the third day only, etc. Therefore, it is completely meaningless to consider the interaction between particles in two different systems. Nevertheless, Jordan and Klein have shown that it is possible to describe a real ensemble of particles which are interacting with each other through a wave field $\Psi(\mathbf{x})$ which actually exists in three-dimensional space if the particles are bosons. According to them, while so far for $V(\mathbf{x})$ in the Hamiltonian

1. Kitaro Nishida (1870–1945) was a renowned Japanese philosopher and writer. He founded his own school of philosophy based on the precept that "pure experience" is the sole reality. His philosophy attempted in his later years to approximate the concept of *Mu* (nothingness) of Zen. Tomonaga's father, Sanjuro (1871–1951) was a contemporary professor with Nishida in the department of philosophy, Kyoto University.

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{x}) \tag{6-20}$$

of field equation (6-1') we considered only potential energy due to the external field, but if there is an interaction between particles, for example, Coulomb repulsion, then we must add to the potential energy $V(\mathbf{x})$ of the external field a potential energy

$$V_{\text{wave}}(\mathbf{x}) = e \int \frac{e\Psi^\dagger(\mathbf{x}')\Psi(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, dv', \tag{6-21}$$

which is caused by the charge density $e\Psi^\dagger(\mathbf{x})\Psi(\mathbf{x})$ of the wave field itself, and use

$$V'(\mathbf{x}) = V(\mathbf{x}) + V_{\text{wave}}(\mathbf{x}).$$

Namely, as a Hamiltonian in (6-1') we use

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{x}) + V_{\text{wave}}(\mathbf{x}) \tag{6-20'}$$

and the equation

$$\left[\frac{1}{2m}\mathbf{p}^2 + V(\mathbf{x}) + e \int \frac{e\Psi^\dagger(\mathbf{x}')\Psi(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, dv' + \frac{\hbar}{i}\frac{\partial}{\partial t}\right]\Psi(\mathbf{x}, t) = 0 \tag{6-22}$$

instead of (6-1'). Jordan and Klein discovered that if we do this, this theory and the quantum theory of a boson system with the Hamiltonian

$$H = \sum_{\nu=1}^{N}\left[\frac{1}{2m}\mathbf{p}_n^2 + V(\mathbf{x})\right] + \sum_{\nu>\nu'}^{N} \frac{e^2}{|\mathbf{x}_\nu - \mathbf{x}_{\nu'}|} \tag{6-23}$$

give results that agree in all respects. It is worth noting that in the second summation of (6-23) the term with $\nu = \nu'$ is excluded. As a result of this, for the case of one particle, the term involving $e^2/|\mathbf{x}_\nu - \mathbf{x}_{\nu'}|$ does not appear in (6-23) even if we use (6-22), in which $V_{\text{wave}}(\mathbf{x})$ is included.

From this work of Jordan and Klein, the difference between Ψ and ψ becomes even clearer. The field equation (6-22) for Ψ has an entirely different form from (6-1), which gives the probability amplitude of one particle, i.e.,

$$\left[\frac{1}{2m}\mathbf{p}^2 + V(\mathbf{x}) + \frac{\hbar}{i}\frac{\partial}{\partial t}\right]\psi(\mathbf{x}, t) = 0. \tag{6-22'}$$

Moreover, this difference is fundamental. The reason is that, according to the transformation theory, the probability amplitude must satisfy the principle of superposition, and therefore the equation satisfied by ψ should always be in linear form, but (6-22) is not linear with respect to Ψ. For this reason even if we consider that Ψ is not a q-number, the field equation (6-22) is of such a nature that it can never be considered as the equation for ψ. We now see clearly that it is entirely incorrect to say, "Ψ is obtained by the second quantization of ψ."

Equation (6-22), although not derivable by heuristic reasoning alone, is a field equation which takes into account the interaction of particles correctly, and if Ψ is not quantized, it should correspond to the classical Maxwell equation. Do you ask, Why then is there an $\hbar$ on the left-hand side of (6-22)? That is a good question. It is the sort of question that a happy-go-lucky person cannot ask. Since you noticed this very good point, try to answer it yourself. (Hint: Replace in the argument $m \to \hbar\hat{m}$, $V \to \hbar\hat{v}$, $e \to \hbar\hat{e}$, and $\Psi \to \hat{\Psi}/\sqrt{\hbar}$. Consider the meaning of these replacements.)

I have already talked for an unintentionally long time about Dirac's acrobatics, but all he wanted to show was that a system of many bosons and the wave field in three-dimensional space are equivalent in quantum mechanics. He tried to apply this conclusion to photons to discuss quantum-mechanically the emission, absorption, and scattering of photons by atoms. You might think that in order to apply the argument used so far directly to photons, we need the equation of probability amplitude of one photon. However, the photon is a relativistic particle, and we cannot use (6-1) for it. Many people at that time attempted without success to find a probability amplitude for one photon. (It was shown later that in the relativistic theory, not only for the case of photons, but in general, the probability amplitude in the x, y, z space does not exist.) However, as has now become clear after this long talk, it is the field equation that has to be quantized and not that of the probability amplitude. Therefore, even if we do not know the equation of probability amplitude, it suffices just to quantize the field equation, if it is known. Dirac knew this right from the beginning without needing this long exposition. He showed that by extending Debye's idea the correct answer for the absorption and emission of photons and the scattering of photons by atoms is obtained by quantizing the radiation field. Following this work of Dirac's, Fermi as well as Heisenberg and Pauli formalized more completely the interaction between an atom and the electromagnetic field by considering the electron and the quantized Maxwell field simultaneously. In particular, in their massive paper titled "On the Quantum Mechanics of the Wave Field" (1929), Heisenberg and Pauli, unlike Dirac and Fermi, treated the problem by considering not only the electromagnetic field but also the electron itself as a quantized field. (In this case, however, quantization cannot be done by (6-14'), for if so, the electron would be a boson. Then how to quantize an electron? I shall talk about this shortly.) In other words, in this paper they consider the Dirac equation not as the equation for the probability amplitude of an electron but as the relativistic field equation for the electron.

If we accept these premises, then probably it is all right to adopt as another possible relativistic field equation the Klein-Gordon equation, which was rejected by Dirac as an unsuitable equation for probability amplitude. Therefore, probably it is not that surprising that on one clear day[2] it occurred to Pauli to consider the Klein-Gordon equation and to quantize it with the Maxwell equations in a similar vein as the Heisenberg-Pauli theory.

Therefore he wrote a paper in 1934, helped by his assistant Victor Weisskopf. The paper's theme, "Nature has no reason to reject particle of spin zero," is the subtitle of my lecture today. Now we really head for the crux of the matter, but before doing that, let me finish the aforementioned problem, namely how to quantize the electron field.

Thus far I have told you that the particle which appears from the quantization of the wave field is a boson, but we would also like to know if we can treat a fermion like the electron by the method of field quantization. It was the work by Jordan and Wigner which appeared in 1928, one year before the work of Heisenberg and Pauli, which answered this question. Their answer was affirmative. However, we cannot, of course, use the commutation relation (6-14'), but instead we must use the relation in which the minus sign on the left-hand side of (6-14') is replaced by a plus sign.

$$\begin{aligned}\Psi(\mathbf{x})\Pi(\mathbf{x}') + \Pi(\mathbf{x}')\Psi(\mathbf{x}) &= i\hbar\delta(\mathbf{x}-\mathbf{x}')\\ \Psi(\mathbf{x})\Psi(\mathbf{x}') + \Psi(\mathbf{x}')\Psi(\mathbf{x}) &= 0 \\ \Pi(\mathbf{x})\Pi(\mathbf{x}') + \Pi(\mathbf{x}')\Pi(\mathbf{x}) &= 0.\end{aligned} \tag{6-14'$_+$}$$

They discovered these relations, which are called *anticommutation relations*. When these relations exist, the eigenvalues of the observable N_n defined by (6-18) are

$$\text{eigenvalues of } N_n = 0, 1, \tag{6-18'$_+$}$$

and therefore either 0 or 1 of this particle can exist in the state n, and thus Pauli's exclusion principle is clearly satisfied. Furthermore, it is possible to prove that the wave field thus quantized is completely equivalent in all respects to the solution in which only the antisymmetric wave function of the particle system with the Hamiltonian (6-16) or (6-23) is adopted, i.e., a system of fermions. Therefore the theme which I told you earlier à la Nishida's philosophy works for both bosons and fermions.

2. Tomonaga uses the phrase *Un bel dì* from the aria of *Madame Butterfly*.

Now we will go into the story of Pauli and Weisskopf, which is the main theme of this lecture. Since I had to lay the groundwork by delving into the huge problem of wave equals particle and particle equals wave, I have nearly run out of time. However, because of this long preparation you probably have understood the background for Pauli and Weisskopf's work, so let me just give you their results.

As I told you earlier, Pauli and Weisskopf tried to quantize the Klein-Gordon equation together with the Maxwell equations using (6-14'). They were able to do so without any inconsistencies, and from the Klein-Gordon field they obtained bosons of mass m and spin 0, and very interestingly they found that the electric charge of this boson can take $+e$ or $-e$, i.e., be either positive or negative. Moreover, not only was this possible, but they also found through calculation that when there exists a photon whose $h\nu$ is larger than $2mc^2$, then its absorption can create a pair of particles with $+e$ and $-e$, and conversely, if there is a $+e/-e$ pair, then they can be annihilated by emitting $h\nu > 2mc^2$.

Furthermore, Pauli and Weisskopf objected to regarding the Dirac equation as the equation of relativistic probability amplitude of one electron although it is first order. According to them, the Dirac equation is also the relativistic *field* equation for the electron and it cannot be considered to be an equation of probability amplitude in x, y, z space. They insisted that a concept like "the probability of a particle to be at $\mathbf{x}$ in space" is meaningless for relativistic particles—be they electrons, photons, or Klein-Gordon particles—and therefore it is meaningless to interpret $\psi(\mathbf{x})$ as the probability amplitude.

One basis for this last declaration is that, if we consider the Dirac equation to be the equation of probability amplitude of an electron, then there appears the peculiar state in which the electron has a negative energy, and the positive-energy electron will fall into the negative-energy state by emitting energy as a result of interaction with the electromagnetic field. If this can happen, many peculiar phenomena which contradict reality will follow. In order to cope with this difficulty, around 1930 Dirac introduced the hypothesis that the vacuum is the state in which all the negative-energy levels are occupied by electrons. Then because of the Pauli principle, the positive-energy electrons cannot fall into negative energy. In Pauli's opinion, if all the negative-energy levels are filled by so many electrons, there will be an infinite number of electrons, and that is already way beyond the one-body problem.

For these reasons Pauli deemed that, just as in the Heisenberg-Pauli paper, we should treat the Dirac equation as a field equation rather than as an equation for probability amplitude. Dirac, apparently, did not care for Pauli taking Dirac's equation as the field equation and proposed in his new theoretical formulation of the many-electron problem published in 1932, which later came to be called the many-time theory, to use the probability amplitude ψ in the coordinate system [however, in order to make ψ relativistic we extend the concept of coordinates in the wave function by assigning a different time for each electron, hence $\psi(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2, \mathbf{x}_3, t_3, \ldots)$].

Now the other side of the story is that Dirac himself was starting to anticipate from his hypothesis of filling the negative-energy levels that in addition to the usual electron, i.e., an electron with negative charge, there exists an electron with positive charge, just as bosons of $+e$ and $-e$ appear from quantizing the Klein-Gordon field. His idea was that if the filling of negative-energy levels is incomplete and an electron is missing from some negative-energy level, then that "hole" has a positive energy and behaves as if it has a positive charge. (Dirac's idea is that lack of negative must mean positive.) Therefore, a hole would look like an electron with positive charge. (Initially, Dirac thought that this hole was a proton. However, Oppenheimer raised the criticism that according to this idea, the hole would be immediately filled by a nearby electron and the hydrogen atom could not exist stably. Furthermore, it was pointed out by the mathematician Hermann Weyl that this hole should behave as if it had the same mass as an electron.)

According to this idea, if there is a photon with $h\nu > 2mc^2$, then an electron in a negative-energy level is excited to a positive-energy level, absorbing energy, and as a result an electron with positive energy and a hole in a negative level—in other words, a positive-energy, positive-charge electron—are created. By the way, this positively charged electron was discovered experimentally in 1932 by Carl Anderson and was named *positron*.

I told you that both positively and negatively charged particles can appear when the Klein-Gordon equation is quantized. However, Dirac already pointed out in his 1928 paper, which I talked about in lecture 3, that even without quantization the equation has one solution which behaves like a particle with negative charge and one which behaves like a particle with positive charge. (I shall discuss this further in the next lecture.) He gave this as one of the reasons for insisting that the electron cannot be described by this equation. (At that time only electrons with negative charge were believed to exist.)

However, after the discovery of the positron, not only did this reason become groundless but it also became apparent that both bosons related to the Klein-Gordon equation and fermions related to the Dirac equation have a very similar property in that they have positive and negative charge, and from these resemblances it became quite natural to believe that both the Klein-Gordon equation and the Dirac equation have equally good raison d'être. For these reasons Pauli confidently pushed for the resurrection of the Klein-Gordon field. Furthermore, if you quantize the Dirac equation in the manner of Heisenberg and Pauli, then it is possible with some ingenuity to incorporate positrons and pair creation into the theory without any artificial postulates such as filling the negative-energy levels and holes. Because of these points, Pauli thought it much more natural to regard the Dirac equation as the field equation rather than as the equation of probability amplitude as Dirac preferred.

Thus Pauli and Weisskopf reinstated the Klein-Gordon equation, which Dirac had rejected. It seems Pauli was quite euphoric about this work, and a part of the paper sounds as if he is teasing Dirac by using exactly the phrase which Dirac had used in his other paper. Freely translated, it reads as follows.

"The most interesting part of our theory is that the energy is always positive automatically (namely, without using a superfluous hypothesis such as hole theory). After witnessing that the relativistic scalar theory (the theory of the Klein-Gordon field) can be constructed without any such hypothesis, one might wonder *why nature had made no use* of the possibility from the theory that there exist spin-zero bosons with charge $\pm e$ and the possibility that they can be created from $h\nu$ or annihilated, emitting $h\nu$." The italicized phrase is taken directly from the paper in which Dirac predicted the existence of a certain particle (magnetic monopole). It seems to me that the tone of this sentence and that of Dirac's sentence (which I quoted in lecture 3) in the beginning of his paper on the Dirac equation are nearly identical.

This sentence sounds as if Pauli wanted to say that there exist charged particles with spin 0, but when we read on we find on the contrary that he sounds as if he is looking for some reason why such a particle cannot be found. However, when Pauli was writing this paper in 1934, the new idea of the meson was already taking shape in Yukawa's mind, and the relativistic scalar theory (more accurately, pseudoscalar theory) would play a big role in the theory of mesons. Therefore, nature indeed did use this possibility in the appearance of π mesons.

All the same, Pauli and Weisskopf's work and Yukawa's idea of the meson appeared one after the other with incredible timing. The history of physics can sometimes be very theatrical.