This is the relation used by Schrödinger* for calculating certain integrals involving Laguerre polynomials.

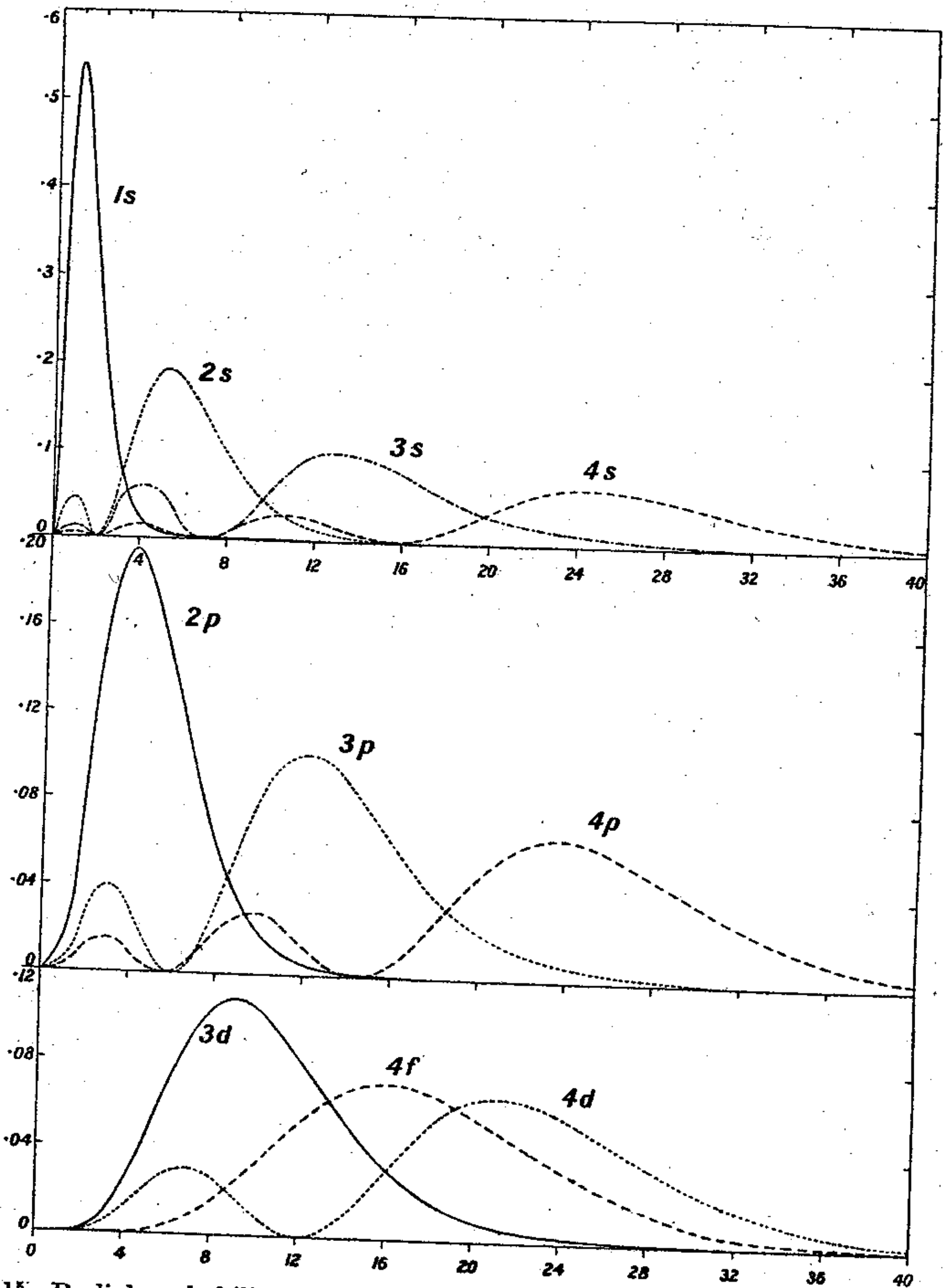

Fig. 1[5]. Radial probability distribution a $R^2(nl)$ for several of the lowest levels in hydrogen. (Abscissa is the radius in atomic units.)

In order to normalize the radial functions we need the result

$$\int_0^\infty \rho^{2l+2} e^{-\rho} [L_{n+l}^{2l+1}(\rho)]^2 d\rho = \frac{2n[(n+l)!]^3}{(n-l-1)!}, \qquad (17)$$

which is readily obtained from the generating function or from Eckart's

* Schrödinger, Ann. der Phys. 80, 485 (1926).

formula. Therefore the final formula for the normalized radial function $R(nl)$ is

$$R(nl)=\sqrt{\frac{Z\,(n-l-1)!}{n^2\mathrm{a}[(n+l)!]^3}}\,e^{-\rho/2}\rho^{l+1}\,L_{n+l}^{2l+1}(\rho),\qquad \rho=\frac{2Zr}{n\mathrm{a}} \tag{18}$$

which is normalized in the sense $\int_0^\infty R^2(nl)\,dr=1$. Several of these functions are given explicitly in Table 1[5]. The probability of finding the electron in dr at r is $R^2(nl)\,dr$; this distribution function is plotted in Fig. 1[5] for some of the lowest states.

TABLE 1[5]. *Normalized radial eigenfunctions for $Z=1$.**

$$R(1s)=-2re^{-r}$$

$$R(2s)=-\frac{1}{\sqrt{2}}re^{-\frac{1}{2}r}(1-\tfrac{1}{2}r)$$

$$R(3s)=-\frac{2}{3\sqrt{3}}re^{-\frac{1}{3}r}(1-\tfrac{2}{3}r+\tfrac{2}{27}r^2)$$

$$R(4s)=-\tfrac{1}{4}re^{-\frac{1}{4}r}(1-\tfrac{3}{4}r+\tfrac{1}{8}r^2-\tfrac{1}{192}r^3)$$

$$R(2p)=-\frac{1}{2\sqrt{6}}r^2e^{-\frac{1}{2}r}$$

$$R(3p)=-\frac{8}{27\sqrt{6}}r^2e^{-\frac{1}{3}r}(1-\tfrac{1}{6}r)$$

$$R(4p)=-\tfrac{1}{16}\sqrt{\tfrac{5}{3}}r^2e^{-\frac{1}{4}r}(1-\tfrac{1}{4}r+\tfrac{1}{80}r^2)$$

$$R(3d)=-\frac{4}{81\sqrt{30}}r^3e^{-\frac{1}{3}r}$$

$$R(4d)=-\frac{1}{64\sqrt{5}}r^3e^{-\frac{1}{4}r}(1-\tfrac{1}{12}r)$$

$$R(4f)=-\frac{1}{768\sqrt{35}}r^4e^{-\frac{1}{4}r}$$

The average values of various powers of r for the hydrogenic wave functions are given in Table 2[5].†

TABLE 2[5].

k	$\mathrm{a}^{-k}\int_0^\infty r^k R^2(nl)\,dr$
1	$\frac{1}{2Z}[3n^2-l(l+1)]$
2	$\frac{n^2}{2Z^2}[5n^2+1-3l(l+1)]$
3	$\frac{n^2}{8Z^3}[35n^2(n^2-1)-30n^2(l+2)(l-1)+3(l+2)(l+1)l(l-1)]$
4	$\frac{n^4}{8Z^4}[63n^4-35n^2(2l^2+2l-3)+5l(l+1)(3l^2+3l-10)+12]$
−1	$\frac{Z}{n^2}$
−2	$\frac{Z^2}{n^3(l+\frac{1}{2})}$
−3	$\frac{Z^3}{n^3(l+1)(l+\frac{1}{2})l}$
−4	$\frac{Z^4\frac{1}{2}[3n^2-l(l+1)]}{n^5(l+\frac{3}{2})(l+1)(l+\frac{1}{2})l(l-\frac{1}{2})}$

* In this table r is measured in atomic units. The general eigenfunctions for any Z and arbitrary length unit are obtained by multiplying the functions of this table by $\sqrt{Z/\mathrm{a}}$ and replacing r by Zr/a.

† The average values of r^{-5} and r^{-6} may be found in VAN VLECK, Proc. Roy. Soc. **A143**, 679 (1934).

The quantities occurring in 7[4]5 are thus completely expressed in terms of the integrals over the radial eigenfunctions.

The calculations may be exemplified by a detailed consideration of the line strengths in the fine structure of H_α, the ensemble of the $n=3\rightarrow n=2$

TABLE 3[5]. *Values of* $\left[\int_0^\infty r\,R(n\,l)\,R(n'\,l-1)\,dr\right]^2$ *in atomic units for* $n'\neq n$.

$n\,l$	$n'\,l-1$	
np	$1s$	$2^8 n^7(n-1)^{2n-5}(n+1)^{-2n-5}$
	$2s$	$2^{17} n^7(n^2-1)(n-2)^{2n-6}(n+2)^{-2n-6}$
	$3s$	$2^8 3^7 n^7(n^2-1)(n-3)^{2n-8}(7n^2-27)^2(n+3)^{-2n-8}$
	$4s$	$2^{26} 3^{-2} n^7(n^2-1)(n-4)^{2n-10}(23n^4-288n^2+768)^2(n+4)^{-2n-10}$
	$5s$	$2^8 3^{-2} 5^9 n^7(n^2-1)(n-5)^{2n-12}(91n^6-2545n^4+20625n^2-46875)^2(n+5)^{-2n-12}$
nd	$2p$	$2^{19} 3^{-1} n^9(n^2-1)(n-2)^{2n-7}(n+2)^{-2n-7}$
	$3p$	$2^{11} 3^9 n^9(n^2-1)(n^2-4)(n-3)^{2n-8}(n+3)^{-2n-8}$
	$4p$	$2^{30} 3^{-1} 5^{-1} n^9(n^2-1)(n^2-4)(n-4)^{2n-10}(9n^2-80)^2(n+4)^{-2n-10}$
	$5p$	$2^{11} 3^{-3} 5^9 n^9(n^2-1)(n^2-4)(n-5)^{2n-12}(67n^4-1650n^2+9375)^2(n+5)^{-2n-12}$
nf	$3d$	$2^{13} 3^9 5^{-1} n^{11}(n^2-1)(n^2-4)(n-3)^{2n-9}(n+3)^{-2n-9}$
	$4d$	$2^{38} 3^{-2} 5^{-1} n^{11}(n^2-1)(n^2-4)(n^2-9)(n-4)^{2n-10}(n+4)^{-2n-10}$
	$5d$	$2^{18} 3^{-2} 5^{11} 7^{-1} n^{11}(n^2-1)(n^2-4)(n^2-9)(n-5)^{2n-12}(11n^2-175)^2(n+5)^{-2n-12}$

TABLE 4[5]. *Values of* $\left[\int_0^\infty r\,R(n\,l)\,R(n'\,l-1)\,dr\right]^2$ *in atomic units.*

	$2p$	$3p$	$4p$	$5p$	$6p$	$7p$	$8p$
$1s$	1·66	0·267	0·093	0·044	0·024	0·015	0·010
$2s$	27·0	9·4	1·64	0·60	0·29	0·17	0·10
$3s$	0·9	162	29·9	5·1	1·9	0·9	0·5
$4s$	0·15	6·0	540	72·6	11·9	5·7	2·1
$5s$	0·052	0·9	21·2	1125	134	41·4	21·8
$6s$	0·025	0·33	2·9		2835		
$7s$	0·014	0·16	1·4			5292	
$8s$	0·009	0·09	0·8				9072

	$3d$	$4d$	$5d$	$6d$	$7d$	$8d$
$2p$	22·52	2·92	0·95	0·44	0·242	0·149
$3p$	101·2	57·2	8·8	3·0	1·44	0·82
$4p$	1·7	432	121·9	19·3	7·7	3·2
$5p$	0·23	9·1	1181·25	203	36	12·3
$6p$	0·08	1·3		2592		
$7p$	0·03	0·5			4961·25	
$8p$	0·02	0·2				8640

	$4f$	$5f$	$6f$	$7f$	$8f$
$3d$	104·6	11·0	3·2	1·4	0·8
$4d$	252·0	197·8	26·9	8·6	3·9
$5d$	2·75	900			
$6d$	0·32		2187		
$7d$	0·08			4410	
$8d$	0·04				7920