Many years ago I wrote the book *Lie Groups, Lie Algebras, and Some of Their Applications* (NY: Wiley, 1974). That was a big book: long and difficult. Over the course of the years I realized that more than 90% of the most useful material in that book could be presented in less than 10% of the space. This realization was accompanied by a promise that some day I would do just that — rewrite and shrink the book to emphasize the most useful aspects in a way that was easy for students to acquire and to assimilate. The present work is the fruit of this promise.

In carrying out the revision I’ve created a sandwich. Lie group theory has its intellectual underpinnings in Galois theory. In fact, the original purpose of what we now call Lie group theory was to use continuous groups to solve differential (continuous) equations in the spirit that finite groups had been used to solve algebraic (finite) equations. It is rare that a book dedicated to Lie groups begins with Galois groups and includes a chapter dedicated to the applications of Lie group theory to solving differential equations. This book does just that. The first chapter describes Galois theory, and the last chapter shows how to use Lie theory to solve some ordinary differential equations. The fourteen intermediate chapters describe many of the most important aspects of Lie group theory and provide applications of this beautiful subject to several important areas of physics and geometry.

Over the years I have profitted from the interaction with many students through comments, criticism, and suggestions for new material or different approaches to old. Three students who have contributed enormously during the past few years are Dr. Jairzinho Ramos-Medina, who worked with me on Chapter 15 (Maxwell’s Equations), and Daniel J. Cross and Timothy Jones, who aided this computer illiterate with much moral and ether support. Finally, I thank my beautiful wife Claire for her gracious patience and understanding throughout this long creation process.
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