13

Contraction

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New Lie groups can be constructed from old by a process called group contraction. Contraction involves reparameterization of the Lie group's parameter space in such a way that the group multiplication properties, or commutation relations in the Lie algebra, remain well defined even in a singular limit. In general, the properties of the original Lie group have well-defined limits in the contracted Lie group. For example, the parameter space for the contracted group is well-defined and noncompact. Other properties with well-defined limits include: Casimir operators; basis states of representations; matrix elements of operators; and Baker–Campbell–Hausdorff formulas. Contraction provides limiting relations among the special functions of mathematical physics. We describe a particularly simple class of contractions — the Inönü–Wigner contractions — and treat one example of a contraction not in this class.

13.1 Preliminaries

It is possible to construct new Lie algebras from old by a certain limiting process called **contraction**. In this process a new set of basis vectors Y_r is related to the initial set of basis vectors X_i through a parameter-dependent change of basis: $Y_r = M_r{}^i(\epsilon)X_i$. The structure constants have the transformation properties of a tensor: $C_{rs}{}^t(\epsilon) = M_r{}^i(\epsilon)M_s{}^j(\epsilon)C_{ij}{}^k(M(\epsilon)^{-1})_k{}^t$ (cf. Eq. 4.22). As long as the change of basis transformation is nonsingular the Lie algebra is unchanged.

If the transformation becomes singular, the structure constants $C_{rs}{}^t(\epsilon)$ may still converge to a well-defined limit. It is often the case that the structure constants

$$C_{rs}{}^t(0) = \lim_{\epsilon\to 0} C_{rs}{}^t(\epsilon) \tag{13.1}$$

exist and define a Lie algebra that is different from the original Lie algebra.

13.2 Inönü–Wigner Contractions

If a Lie algebra $\mathfrak{g}$ has a subalgebra $\mathfrak{h}$ and a complementary subspace $\mathfrak{p}$ with commutation relations of the form

$$\begin{array}{rcll} \mathfrak{g} & = & \mathfrak{h}+\mathfrak{p} & \\ [\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h} & \text{subalgebra} \\ [\mathfrak{h},\mathfrak{p}] & \subseteq & \mathfrak{p} & \text{this is important} \\ [\mathfrak{p},\mathfrak{p}] & \subseteq & \mathfrak{h}+\mathfrak{p} & \text{this is always true} \end{array} \tag{13.2}$$

then the Inönü–Wigner contraction of $\mathfrak{g} \to \mathfrak{g}'$ involves the following change of basis transformation

$$\begin{bmatrix} \mathfrak{h}' \\ \mathfrak{p}' \end{bmatrix} = \begin{bmatrix} I_{\dim(\mathfrak{h})} & 0 \\ 0 & \epsilon\, I_{\dim(\mathfrak{p})} \end{bmatrix} \begin{bmatrix} \mathfrak{h} \\ \mathfrak{p} \end{bmatrix} \tag{13.3}$$

where $\dim(\mathfrak{h})$ is the dimension of the subalgebra $\mathfrak{h}$. The commutation relations of $\mathfrak{g}'$ are well-defined for all values of ϵ, including the singular limit $\epsilon \to 0$:

$$\begin{array}{rcccccccl} [\mathfrak{h}',\mathfrak{h}'] & = & [\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h} & & & & \\ [\mathfrak{h}',\mathfrak{p}'] & = & [\mathfrak{h},\epsilon\mathfrak{p}] & = & \epsilon\,[\mathfrak{h},\mathfrak{p}] & \lim_{\epsilon\to 0} & \epsilon\mathfrak{p} & \to & \mathfrak{p}' \\ [\mathfrak{p}',\mathfrak{p}'] & = & [\epsilon\mathfrak{p},\epsilon\mathfrak{p}] & = & \epsilon^2\,[\mathfrak{p},\mathfrak{p}] & \lim_{\epsilon\to 0} & \epsilon^2(\mathfrak{h}+\mathfrak{p}) & \to & 0 \end{array} \tag{13.4}$$

In the limit $\epsilon \to 0$ the contracted algebra $\mathfrak{g}'$ is the semidirect sum of the original subalgebra $\mathfrak{h}$ and the subalgebra $\mathfrak{p}'$, where $\mathfrak{p}'$ is commutative and $[\mathfrak{h},\mathfrak{p}'] \subseteq \mathfrak{p}'$:

$$\begin{array}{lcl} \mathfrak{g} & = & \mathfrak{h}+\mathfrak{p} \\ [\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h} \\ [\mathfrak{h},\mathfrak{p}] & \subseteq & \mathfrak{p} \\ [\mathfrak{p},\mathfrak{p}] & \subseteq & \mathfrak{h}+\mathfrak{p} \end{array} \qquad \mathfrak{p}\to\mathfrak{p}'=\epsilon\mathfrak{p} \qquad \longrightarrow \qquad \begin{array}{lcl} \mathfrak{g}' & = & \mathfrak{h}+\mathfrak{p}' \\ [\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h} \\ [\mathfrak{h},\mathfrak{p}'] & \subseteq & \mathfrak{p}' \\ [\mathfrak{p}',\mathfrak{p}'] & = & 0 \end{array} \tag{13.5}$$

13.3 Simple Examples of Inönü–Wigner Contractions

In this section we illustrate several facets of Inönü–Wigner contractions by contracting three different orthogonal groups.

13.3.1 The Contraction $SO(3) \to ISO(2)$

The infinitesimal generators of the Lie group $SO(3)$ may be chosen as $L_1 = X_{23} = x_2\partial_3 - x_3\partial_2 = \epsilon_{1jk}x_j\partial_k$, with L_2 and L_3 defined by cyclic permutation. The commutation relations are

$$\begin{array}{lcl} [L_1, L_2] & = & -L_3 \\ [L_2, L_3] & = & -L_1 \\ [L_3, L_1] & = & -L_2 \end{array} \tag{13.6}$$

Under contraction with respect to the subalgebra of rotations about the z-axis (infinitesimal generator L_3) the operators L_1 and L_2 go to

$$\begin{bmatrix} \epsilon L_1 \\ \epsilon L_2 \end{bmatrix} \overset{\epsilon=1/R}{\longrightarrow} \begin{bmatrix} (1/R)L_1 \\ (1/R)L_2 \end{bmatrix} \to \begin{bmatrix} -P_2 \\ +P_1 \end{bmatrix} \tag{13.7}$$

The commutation relations of the contracted algebra, $ISO(2) = E(2)$, are

$$\begin{array}{lcl} [L_3, P_1] & = & -P_2 \\ [L_3, P_2] & = & +P_1 \\ [P_1, P_2] & = & 0 \end{array} \tag{13.8}$$

The three operators L_3, P_1, P_2 generate the group of Euclidean motions of the plane, $E(2)$, or inhomogeneous orthogonal transformations in the plane R^2, $ISO(2)$. This group consists of rotations about the z-axis, generated by L_3, and displacements of the origin in the x- and y- directions, generated by $P_1 = \partial_1$ and $P_2 = \partial_2$.

To verify this interpretation we can imagine the group $SO(3)$ acting on the sphere $x^2 + y^2 + z^2 = R^2$ in the neighborhood of the north pole

$(0, 0, R)$, as shown in Fig. 13.1. An element in the Lie algebra $\mathfrak{so}(3)$ can be written in the form

$$\theta_1 L_1 + \theta_2 L_2 + \theta_3 L_3 \longrightarrow (-d_2)\left(\frac{L_1}{R}\right) + (+d_1)\left(\frac{L_2}{R}\right) + \theta_3 L_3 \quad (13.9)$$

In the limit $R \to \infty$ we find

$$\begin{aligned} \tfrac{1}{R}L_1 = \tfrac{1}{R}(x^2\partial_3 - x^3\partial_2) = \tfrac{1}{R}(y\partial/\partial z - R\partial/\partial y) \quad &\to \quad -\partial/\partial y = -\partial_2 = -P_2 \\ \tfrac{1}{R}L_2 = \tfrac{1}{R}(x^3\partial_1 - x^1\partial_3) = \tfrac{1}{R}(R\partial/\partial x - x\partial/\partial z) \quad &\to \quad +\partial/\partial x = +\partial_1 = +P_1 \end{aligned} \quad (13.10)$$

The contracted limits of the operators L_1 and L_2 in the limit of a sphere of very large radius are operators $-P_2, +P_1$ describing displacements in the $-y$ and $+x$ directions. In addition, the parameters θ_1, θ_2 and d_1, d_2 are related by

$$\begin{aligned} d_1 &= +R\theta_2 \\ d_2 &= -R\theta_1 \end{aligned} \quad (13.11)$$

As the radius of the sphere becomes very large, the two angles θ_1, θ_2 become small with the product $R\theta_i$ $(i = 1, 2)$ approaching a well-defined limit. This corresponds to a rotation through an angle $\theta_2 = d_1/R$ about the y-axis producing a displacement of d_1 in the x-direction, and a rotation through an angle $\theta_1 = d_2/R$ about the x-axis producing a displacement of $-d_2$ in the y-direction.

The Casimir operator for the group $SO(3)$ contracts to an invariant operator as follows:

$$\begin{aligned} \mathcal{C}^2[SO(3)] &= L_1^2 + L_2^2 + L_3^3 \\ \mathcal{C}^2[ISO(2)] &= \lim(1/R^2)\ \mathcal{C}^2[SO(3)] \\ &= \lim[(L_1/R)^2 + (L_2/R)^2 + (L_3/R)^2] \\ &= (-P_2)^2 + (+P_1)^2 + 0 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \end{aligned} \quad (13.12)$$

This is just the Laplacian operator on the plane R^2.

13.3.2 The Contraction $SO(4) \to ISO(3)$

This group is similar to $SO(3)$ and can be treated similarly. The six generators are

$$\begin{aligned} L_i &= \epsilon_{ijk} x_j \partial_k \\ V_i &= x_i\partial_4 - x_4\partial_i \qquad 1 \le i \ne j \ne k \le 3 \end{aligned} \quad (13.13)$$

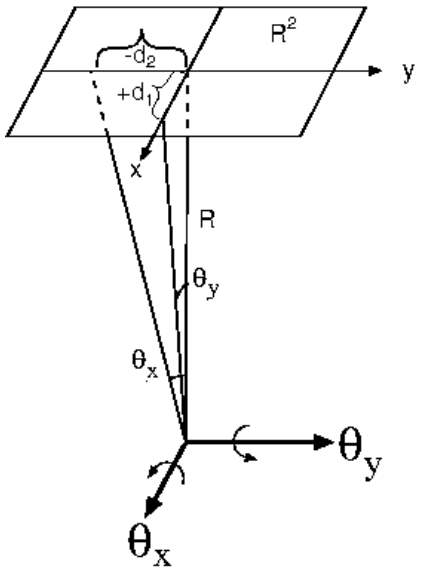

Fig. 13.1. Rotations on the surface of a sphere of radius R approach displacements in the plane as $R \to \infty$.

The commutation relations are

$$\begin{aligned} [L_i, L_j] &= -\epsilon_{ijk} L_k \\ [L_i, V_j] &= -\epsilon_{ijk} V_k \\ [V_i, V_j] &= -\epsilon_{ijk} L_k \end{aligned} \tag{13.14}$$

We contract with respect to the subgroup $SO(3)$ generated by the angular momentum operators L_i, defining

$$-P_i = \lim_{R\to\infty} \frac{1}{R} V_i = \lim_{R\to\infty} \frac{1}{R}(x_i \partial_4 - x_4 \partial_i) = -\partial_i \tag{13.15}$$

The commutation relations of the contracted algebra are

$$\begin{aligned} [L_i, L_j] &= -\epsilon_{ijk} L_k \\ [L_i, P_j] &= -\epsilon_{ijk} P_k \\ [P_i, P_j] &= 0 \end{aligned} \tag{13.16}$$

The operators P_i describe displacements in the x-, y-, and z- ($i = 1, 2, 3$) directions. The contracted group is $ISO(3)$, the Euclidean, or inhomogeneous orthogonal group, on R^3.

As in the case $SO(3) \to ISO(2)$, we can contract the second order Casimir operator of $SO(4)$ to that of $ISO(3)$

$$\begin{aligned} \mathcal{C}_1^2[ISO(3)] &= \lim_{R\to\infty}(1/R^2)(\mathbf{L}\cdot\mathbf{L}+\mathbf{V}\cdot\mathbf{V}) \\ &= \lim_{R\to\infty}[(\mathbf{L}/R)\cdot(\mathbf{L}/R)+(\mathbf{V}/R)\cdot(\mathbf{V}/R)] \quad (13.17) \\ &= 0+\mathbf{P}\cdot\mathbf{P}=\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \quad (13.18) \end{aligned}$$

As before, this is no surprise. The contracted operator is the Laplacian on R^3. What is a surprise is that there is a second nontrivial invariant operator. For $SO(4)$ this is [cf. Eq. (9.24)]

$$\mathcal{C}_2^2[SO(4)]=\epsilon^{ijkl}X_{ij}X_{kl}\to 8\mathbf{L}\cdot\mathbf{V} \quad (13.19)$$

The contracted limit of this operator is

$$\begin{aligned} \mathcal{C}_2^2[ISO(3)]/8 &= \lim_{R\to\infty}(1/R)(\mathbf{L}\cdot\mathbf{V}) \\ &= \lim_{R\to\infty}[\mathbf{L}\cdot(\mathbf{V}/R)]=-\mathbf{L}\cdot\mathbf{P} \quad (13.20) \end{aligned}$$

The two invariant operators $\mathbf{P}\cdot\mathbf{P}=\nabla^2$ and $\mathbf{L}\cdot\mathbf{P}=-\mathbf{L}\cdot\nabla$ form a complete set of invariant operators for the group $ISO(3)$.

13.3.3 The Contraction $SO(4,1)\to ISO(3,1)$

The group $ISO(3,1)$ consists of proper Lorentz transformations $[SO(3,1)]$ that leave invariant the quadratic form

$$x^2+y^2+z^2-(ct)^2 \quad (13.21)$$

as well as displacements of the origin in the three spacelike directions and one timelike direction. The inhomogeneous Lorentz group, or Poincaré group, leaves invariant space-time intervals

$$(x-x')^2+(y-y')^2+(z-z')^2-(ct-ct')^2 \quad (13.22)$$

This group can be contracted from either $SO(4,1)$ or $SO(3,2)$.

We choose as infinitesimal generators for the group $SO(4,1)$ the operators

$$\begin{array}{lcllll} X_{ij} & = & x_i\partial_j-x_j\partial_i=\epsilon_{ijk}L_k & 1\le i,j,k\le 3 & & \text{rotations} \\ B_{i4} & = & x_i\partial_4+x_4\partial_i & 1\le i\le 3 & & \text{boosts} \\ T_{i5} & = & x_i\partial_5\pm x_5\partial_i & i=1,2,3 & -\text{sign} & \text{space displacements} \\ & & & i=4 & +\text{sign} & \text{time displacements} \end{array} \quad (13.23)$$

This set of generators is contracted with respect to the subgroup $SO(3,1)$ generated by rotations and boosts.

The second order Casimir invariant for $SO(4,1)$ and its contraction to the second order Casimir invariant for the Poincaré group are

$$\begin{aligned} \mathcal{C}^2[SO(4,1)] &= \mathbf{L}\cdot\mathbf{L} - \mathbf{B}\cdot\mathbf{B} + \mathbf{T}\cdot\mathbf{T} - T_{45}^2 \\ \mathcal{C}^2[ISO(3,1)] &= 0 - 0 + \nabla\cdot\nabla - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \end{aligned}$$

However, $SO(4,1)$ has a second Casimir operator, since it is a real form for the rank-two root space B_2. This is a fourth-degree operator that is derived by analytic continuation from the fourth-order Casimir operastor of $SO(5)$ [cf. Eq. (9.22) and (9.23)]

$$\begin{aligned} \mathcal{C}^4[SO(5)] &= W^\alpha W_\alpha \\ W^\alpha &= \epsilon^{\alpha\beta\gamma\mu\nu} X_{\beta\gamma} X_{\mu\nu} \end{aligned} \tag{13.24}$$

where $\epsilon^{\alpha\beta\gamma\mu\nu}$ is the Levi-Civita symbol (antisymmetric tensor) on five symbols, and W_α is similarly defined. The contracted limit of W^α is nonzero only if one of the four remaining symbols (e.g. ν) is 5:

$$\begin{aligned} \lim_{R\to\infty}(1/R)W^\alpha &\to \lim_{R\to\infty}\epsilon^{\alpha\beta\gamma\mu 5}X_{\beta\gamma}[(1/R)X_{\mu 5}] \\ &= \epsilon^{\alpha\beta\gamma\mu 5}X_{\beta\gamma}(\partial/\partial x^\mu) \end{aligned} \tag{13.25}$$

The four vector $\epsilon^{\alpha\beta\gamma\mu 5}X_{\beta\gamma}(\partial/\partial x^\mu)$ is fairly complicated. Since $W^\alpha W_\alpha$ is invariant, it is convenient to compute it for a particle of mass m in a frame in which the particle is at rest

$$P_\mu = (0,0,0,mc) \tag{13.26}$$

In this frame

$$W^\alpha = \epsilon^{\alpha\beta\gamma\mu}X_{\beta\gamma}mc = L_\alpha mc \tag{13.27}$$

Therefore the invariant is

$$W^\alpha W_\alpha = (\mathbf{L}\cdot\mathbf{L})(\mathbf{P}\cdot\mathbf{P}) \tag{13.28}$$

with $\mathbf{P}\cdot\mathbf{P} = \sum P_\mu P^\mu = -(mc)^2$.

It should be emphasized that if an operator is an invariant and its spectrum or interpretation is desired, the operator should be viewed from the coordinate system which most simplifies its determination (principle of maximum laziness).

13.4 The Contraction $U(2) \to H_4$

In this section we consider a group contraction that is not of Inönü-Wigner type. This is the contraction of the compact unitary group $U(2)$ to the solvable group H_4. This contraction relates the angular momentum operators to the single mode photon operators. These are the infinitesimal generators of the groups $U(2)$ and H_4, respectively. This contraction leads to a number of useful relations that are explored in successive sections.

13.4.1 Contraction of the Algebra

The Lie algebra $\mathfrak{u}(2)$ is spanned by infinitesimal generators $J_3, J_\pm, J_0$ with commutation relations

$$\begin{aligned} [J_3, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= 2J_3 \\ [J_0, \mathbf{J}] &= 0 \end{aligned} \tag{13.29}$$

The operators $h_3, h_\pm, h_0$ are related to $J_3, J_\pm, J_0$ by the following change of basis

$$\begin{bmatrix} h_+ \\ h_- \\ h_3 \\ h_0 \end{bmatrix} = \begin{bmatrix} c & & & \\ & c & & \\ & & 1 & \frac{1}{2c^2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} J_+ \\ J_- \\ J_3 \\ J_0 \end{bmatrix} \tag{13.30}$$

These operators satisfy the following commutation relations

$$\begin{aligned} [h_3, h_\pm] &= \pm h_\pm \\ [h_+, h_-] &= 2c^2 h_3 - h_0 \\ [h_0, \mathbf{h}] &= 0 \end{aligned} \tag{13.31}$$

In the limit $c \to 0$ the change of basis transformation become singular but the commutation relations [Eq (13.31)] converge to a well-defined limit satisfied by the single mode photon operators

$$\begin{bmatrix} h_3 \\ h_+ \\ h_- \\ h_0 \end{bmatrix} \xrightarrow{c \to 0} \begin{bmatrix} \hat{n} + \frac{1}{2} I = \frac{1}{2} \left\{ a, a^\dagger \right\} \\ a^\dagger \\ a \\ I \end{bmatrix} \tag{13.32}$$

13.4.2 Contraction of the Casimir Operators

The group $U(2)$ has rank two. Its two Casimir operators are of first and second order

$$\begin{aligned} \mathcal{C}^1 &= J_0 \\ \mathcal{C}^2 &= J_3^2 + \frac{1}{2}(J_+J_- + J_-J_+) \end{aligned} \tag{13.33}$$

Under contraction $J_0 \to h_0$ but the second Casimir operator has a more interesting limit

$$\begin{aligned} \lim_{c\to 0} c^2\mathcal{C}^2 &= \lim_{c\to 0} c^2(h_3 - \frac{1}{2c^2}h_0)^2 + \frac{1}{2}\left[(cJ_+)(cJ_-) + (cJ_-)(cJ_+)\right] \\ &= \lim_{c\to 0} c^2 h_3^2 - \frac{1}{2}(h_3h_0 + h_0h_3) + c^2(-\frac{1}{2c^2}h_0)^2 + \\ &\qquad \frac{1}{2}\left[(h_+)(h_-) + (h_-)(h_+)\right] \end{aligned} \tag{13.34}$$

The operator $(h_0/2c)^2$ is proportional to the square of the first Casimir operator. It therefore commutes with all elements in the Lie algebra. Therefore the remaining set of operators on the right hand side of (13.34) must also commute with all operators in the Lie algebra. In the limit $c \to 0$, $(ch_3)^2 \to 0$ and the remaining operators go to a well-defined limit

$$\lim_{c\to 0} c^2\mathcal{C}^2[U(2)] - (h_0/2c)^2 \to \mathcal{C}^2[H_4] = -\frac{1}{2}\left[(\hat{n} + \frac{1}{2}I)I + I(\hat{n} + \frac{1}{2}I)\right] + \frac{1}{2}(aa^\dagger + a^\dagger a) \tag{13.35}$$

This is a quadratic operator in the generators $\hat{n} + \frac{1}{2}I, a^\dagger, a$, and I of H_4. The value of this operator in the standard Fock space spanned by the photon number states $|0\rangle, |1\rangle, |2\rangle, \cdots$ is zero. It is the other "invisible invariant" for H_4.

13.4.3 Contraction of the Parameter Space

An arbitrary element in the Lie algebra $\mathfrak{u}(2)$ and its counterpart in the algebra $\mathfrak{h}_4$ with basis $h_3, h_\pm, h_0$ is [1, 31]

$$i\theta_\mu J_\mu = \frac{1}{2}\theta e^{-i\phi}J_+ - \frac{1}{2}\theta e^{+i\phi}J_- + i\theta_3 J_3 + i\theta_0 J_0 =$$

$$\frac{\theta}{2c}e^{-i\phi}h_+ - \frac{\theta}{2c}e^{+i\phi}h_- + i\theta_3 h_3 + i(\theta_0 - \frac{\theta_3}{2c^2})h_0 \tag{13.36}$$

In the limit $c \to 0$ the parameter θ must approach zero so that the limits

$$\begin{aligned} \lim_{c\to 0} +\frac{\theta}{2c} e^{-i\phi} &\to +\alpha \\ \lim_{c\to 0} -\frac{\theta}{2c} e^{+i\phi} &\to -\alpha^* \end{aligned} \tag{13.37}$$

exist. In addition, θ_0 should diverge so that $\theta_0 - \frac{\theta_3}{2c^2}$ remains well defined.

13.4.4 Contraction of Representations

The action of the operators h_3 on the angular momentum state $|J, M\rangle$ is

$$h_3 \left| \begin{array}{c} J \\ M \end{array} \right\rangle = (J_3 + \frac{1}{2c^2} J_0) \left| \begin{array}{c} J \\ M \end{array} \right\rangle = (M + \frac{1}{2c^2}) \left| \begin{array}{c} J \\ M \end{array} \right\rangle \tag{13.38}$$

It is useful to measure states from the "lowest" state $|J, -J\rangle$ in the angular momentum multiplet. The state with the quantum number M is the ground state if $M = -J$, and the nth state when

$$n = J + M \tag{13.39}$$

In order for the action of h_3 on $|J, M\rangle$ to have a well-defined limit, we insist that

$$\lim_{c\to 0}(M + \frac{1}{2c^2}) = \lim_{c\to 0}(n - J + \frac{1}{2c^2}) \tag{13.40}$$

be well defined. This is the case when we go through a sequence of larger and larger representations J of dimension $(2J+1)$ as c becomes smaller and smaller. Specifically, we require c and J to be related by [1, 31]

$$\lim_{c\to 0}(-J + \frac{1}{2c^2}) = 0 \quad \text{implies} \quad 2Jc^2 = 1 \tag{13.41}$$

In this case

$$\lim_{\substack{c \to 0 \\ J \to \infty}} h_3 \left| \begin{array}{c} J \\ M \end{array} \right\rangle = n \left| \begin{array}{c} \infty \\ n \end{array} \right\rangle \tag{13.42}$$

13.4.5 Contraction of Basis States

The basis states $|J, M\rangle$ for an angular momentum multiplet are constructed by applying the angular momentum shift up operator $n = J+M$

times to the ground state $|J,-J\rangle$. These states are contracted to the harmonic oscillator states as follows

$$\begin{aligned} |\begin{array}{c} J \\ M=-J+n \end{array}\rangle &= \frac{(J_+)^n}{[(2J)!n!/(2J-n)!]^{1/2}} |\begin{array}{c} J \\ -J \end{array}\rangle \\ |\begin{array}{c} \infty \\ n \end{array}\rangle &= \lim_{J\to\infty} \frac{(cJ_+)^n}{[(2Jc^2)^n n!]^{1/2}} |\begin{array}{c} \infty \\ 0 \end{array}\rangle \\ &= \frac{(a^\dagger)^n}{\sqrt{n!}} |\begin{array}{c} \infty \\ 0 \end{array}\rangle \end{aligned} \tag{13.43}$$

13.4.6 Contraction of Matrix Elements

The matrix elements of the angular momentum operators on the angular momentum basis states contract readily to the matrix elements of the photon operators on the Fock states

$$\begin{gathered} a^\dagger a|\begin{array}{c} \infty \\ n \end{array}\rangle = \lim_{c\to 0}(J_3+\tfrac{1}{2c^2})|\begin{array}{c} J \\ M \end{array}\rangle = \\ \lim_{c\to 0}[J+M+(\tfrac{1}{2c^2}-J)]|\begin{array}{c} J \\ M=n-J \end{array}\rangle \to (n+0)|\begin{array}{c} \infty \\ n \end{array}\rangle \end{gathered} \tag{13.44}$$

$$\begin{gathered} a^\dagger|\begin{array}{c} \infty \\ n \end{array}\rangle = \lim_{c\to 0} cJ_+|\begin{array}{c} J \\ M \end{array}\rangle = \\ \lim_{c\to 0}|\begin{array}{c} J \\ M+1 \end{array}\rangle\sqrt{(J-M)(J+M+1)c^2} \to \sqrt{n+1}\,|\begin{array}{c} \infty \\ n+1 \end{array}\rangle \end{gathered} \tag{13.45}$$

$$\begin{gathered} a|\begin{array}{c} \infty \\ n \end{array}\rangle = \lim_{c\to 0} cJ_-|\begin{array}{c} J \\ M \end{array}\rangle = \\ \lim_{c\to 0}|\begin{array}{c} J \\ M-1 \end{array}\rangle\sqrt{(J+M)(J-M+1)c^2} \to \sqrt{n}|\begin{array}{c} \infty \\ n-1 \end{array}\rangle \end{gathered} \tag{13.46}$$

13.4.7 Contraction of BCH Formulas

Baker–Campbell–Hausdorff formulas, which can easily be derived for $U(2)$ in its faithful 2×2 matrix representation, can readily be contracted to BCH formulas for H_4, which can be derived with only a little more difficulty in its faithful 3×3 matrix representation [cf. Eq. (7.36)]. For example, the following BCH formula for $U(2)$

$$e^{(\zeta J_+ - \zeta^* J_-)} = e^{\tau J_+} e^{\ln(1+\tau^*\tau)J_3} e^{-\tau^* J_-} \qquad \frac{\zeta}{|\zeta|}\tan|\zeta| = \tau \tag{13.47}$$

contracts under $\lim_{c\to 0} \zeta/c \to \alpha$ to the BCH formula for H_4

$$e^{(\alpha a^\dagger - \alpha^* a)} = e^{\alpha a^\dagger} e^{-\frac{1}{2}\alpha^*\alpha I} e^{-\alpha^* a} \qquad \alpha = \lim_{c\to 0} \zeta/c \tag{13.48}$$

13.4.8 Contraction of Special Functions

Special functions that are associated with the group $SU(2)$ include Jacobi polynomials, the associated Legendre polynomials and spherical harmonics, and the Legendre polynomials. The special functions associated with the "harmonic oscillator" group H_4 are the Hermite polynomials and the harmonic oscillator wavefunctions. One might reasonably expect that the Hermite polynomials and harmonic oscillator wavefunctions are related to the Jacobi or associated Legendre polynomials in some contraction limit. This is so.

The spherical harmonics $Y_m^l(\theta,\phi)$ and associated Legendre polynomials $P_m^l(\cos\theta)$ are related by [1, 31]

$$Y_m^l(\theta,\phi) = \frac{e^{im\phi}}{\sqrt{2\pi}} P_m^l(\cos\theta) \qquad Y_{-m}^l(\theta,\phi) = (-)^m Y_{+m}^l(\theta,\phi)^* \tag{13.49}$$

The associated legendre polynomials are defined by

$$P_m^l(u) = (-)^{l+m} \frac{1}{2^l l!} \sqrt{\frac{2l+1}{2}} \sqrt{\frac{(l-m)!}{(l+m)!}} (1-u^2)^{+m/2} \frac{d^{l+m}}{du^{l+m}} (1-u^2)^l \tag{13.50}$$

These polynomials are contracted to harmonic oscillator wavefunctions under $u \to x/\sqrt{l}$ and $l+m=n$:

$$\lim_{c\to 0} l^{-1/4} P_m^l(u = x/\sqrt{l}) =$$

$$\lim_{c\to 0} (-)^n \sqrt{\frac{(2l)! l!^{1/2}}{2^{(2l)} l! l!}} \sqrt{\frac{1}{2^n n! (2lc^2)^n}} [1-2c^2x^2]^{(-1/2c^2)/2} \frac{d^n}{dx^n} [1-2c^2x^2]^{1/2c^2} \tag{13.51}$$

The limit is taken as $c \to 0, l \to \infty, l+m=n, 2lc^2 = 1$. The limit inside the first square root is $1/\sqrt{\pi}$, that within the second is $(2^n n!)^{-1}$. The result of this contraction is

$$\lim_{c\to 0} l^{-1/4} P_m^l(u = x/\sqrt{l}) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{x^2/2} \left(-\frac{d}{dx}\right)^n e^{-x^2} = \psi_n(x) \tag{13.52}$$

where $\psi_n(x)$ is the appropriately normalized harmonic oscillator eigenfunction

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2} \tag{13.53}$$

and $H_n(x)$ is the nth Hermite polynomial.

Under contraction the orthogonality relations obeyed by the associated Legendre functions go over to the orthogonality relations for the harmonic oscillator eigenfunctions

$$\delta_{mm'} = \int_{-1}^{+1} P_m^l(u) P_{m'}^l(u) du \rightarrow$$

$$\lim_{l\to\infty} \int_{-\sqrt{l}}^{+\sqrt{l}} \left(\frac{1}{l^{1/4}} P_m^l(x/\sqrt{l}) \right) \left(\frac{1}{l^{1/4}} P_{m'}^l(x/\sqrt{l}) \right) d(u\sqrt{l}) \rightarrow$$

$$\int_{-\infty}^{+\infty} \psi_n(x) \psi_{n'}(x) dx = \delta_{nn'} \tag{13.54}$$

Unfortunately, it is not possible to derive the completeness relations for the harmonic oscillator eigenfunctions from the completeness relations for the Jacobi or associated Legendre polynomials. However, there is a very simple and beautiful proof of the completeness relations for all special functions associated with compact Lie groups. It is due to Wigner and Stone.

13.5 Conclusion

Contraction of groups to form inequivalent groups can be carried out whenever a singular change of basis can be constructed under which the structure constants have a well-defined limit. Contraction is a particularly useful way to construct nonsemisimple Lie groups from simple and semisimple Lie groups. The contracted group is always noncompact. Contraction of groups provides many useful relations between the original group and its contracted limit. These involve the commutation relations in the Lie algebra, the range of values in the parameter spaces that map onto the groups, the Casimir operators, the basis states of representations, operator matrix elements, Baker–Campbell–Hausdorff formulas, and limiting relations among special functions. These relations have all been illustrated by example.

13.6 Problems

1. Under the contraction $SO(3) \to ISO(2)$ the representations of $SO(3)$ contract to representations of $ISO(2)$. Since $ISO(2)$ is a noncompact group it has no faithful finite-dimensional unitary representations. We therefore consider the following limit

$$\begin{array}{ccccccccc} \lim a & \downarrow & 0 & aJ_\pm & \to & P_\pm & a^2 l(l+1) & \to & p^2 \text{ finite} \\ l & \uparrow & \infty & J_3 & \to & P_3 & \left| \begin{matrix} l \\ m \end{matrix} \right\rangle & \to & \left| \begin{matrix} p \\ m \end{matrix} \right\rangle \end{array}$$

$$(p/a)\beta = l\beta = x \text{ finite}$$

a. Compute the matrix elements of the operators $P_\pm$ in the algebra $\mathfrak{iso}(2)$ and show

$$\left\langle \begin{matrix} l \\ m' \end{matrix} \right| |aJ_\pm| \left. \begin{matrix} l \\ m \end{matrix} \right\rangle \quad \xrightarrow{\lim} \quad \left\langle \begin{matrix} p \\ m' \end{matrix} \right| |P_\pm| \left. \begin{matrix} p \\ m \end{matrix} \right\rangle$$

$$a\sqrt{(l \mp m)(l \pm m + 1)}\ \delta_{m',m\pm 1} \quad \xrightarrow{\lim} \quad p\ \delta_{m',m\pm 1}$$

b. Compute the contracted limit of the Jacobi polynomials and show that

$$\lim P^l_{mn}\left(\cos(x/l)\right) = (-)^{m-n} J_{m-n}(x)$$

where $J_k(x)$ is the kth Bessel function [1, 31].

c. Contract the spherical harmonics and show that

$$\lim \sqrt{\frac{2\pi}{l}} Y^l_m(\beta = x/l) \to J_m(x)$$

d. Contract the Legendre polynomials and show that

$$\lim P^l\left(\cos(\beta = x/l)\right) \to J_0(x)$$

e. In the generating function expression

$$e^{\alpha J_+} Y^l_m(\theta, \phi) = \sum_{k \geq 0} A^l_k Y^l_{m+k}(\theta, \phi) = Y^l_m(\theta', \phi')$$

compute the coefficients A^l_k and the arguments θ', ϕ' explicitly. Contract these results to construct the classical generating functions for Bessel functions.

f. Show that the operator $\mathbf{L} \cdot \mathbf{L}$ contracts to ∇^2 in the plane.

g. Show that the Casimir invariant operator for $SO(3)$ becomes the Laplace-Beltrami operator on $S^2 = SO(3)/SO(2)$ when restricted to the sphere surface, and this operator contracts to the Bessel equation.

2. Under the contraction $\mathfrak{u}(2) \to \mathfrak{h}_4$ the representations of the unitary group $U(2)$ contract to representations of the noncompact Heisenberg group H_4. Since H_4 is noncompact it has no faithful finite dimensional unitary irreducible representations. We therefore contract through a series of representations of $U(2)$ of ever increasing dimensions, as follows:

$$\begin{array}{ccc} \lim \epsilon \to \infty & \epsilon J_\pm \to h_\pm & 2j\epsilon^2 \to 1 \\ j \to +\infty, m \to -\infty & J_3 + \frac{1}{2\epsilon^2} \to h_3 & \mid \begin{matrix} j \\ m \end{matrix} \rangle \to \mid \begin{matrix} \infty \\ n \end{matrix} \rangle \\ j + m = n \text{ (finite)} & \theta \to \frac{\pi}{2} - \sqrt{2}\epsilon x & \end{array}$$

a. Compute the matrix elements

$$\langle \begin{matrix} j \\ m' \end{matrix} \mid \epsilon J_\pm \mid \begin{matrix} j \\ m \end{matrix} \rangle \xrightarrow{\lim} \langle \begin{matrix} \infty \\ n' \end{matrix} \mid h_\pm \mid \begin{matrix} \infty \\ n \end{matrix} \rangle$$

$$\epsilon\sqrt{(j \mp m)(j \pm m + 1)}\ \delta_{m',m\pm1} \xrightarrow{\lim} \begin{matrix} \sqrt{n+1}\ \delta_{n',n+1} \\ \sqrt{n}\ \delta_{n',n-1} \end{matrix}$$

b. Contract the spherical harmonics and show

$$l^{1/4} P^l_{n-l,0}\left(\frac{\pi}{2} - \sqrt{2}\epsilon x\right) \xrightarrow{\lim} \psi_n(x) = N_n H_n(x) e^{-x^2/2}$$

where $\psi_n(x)$ is the nth excited state wavefunction for the harmonic oscillator, $H_n(x)$ is the nth Hermite polynomial, and N_n is the usual normalization coefficient, $N_n = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}}$.

c. Carry out steps **c. - f.** of the previous problem. The results are obtained by making the following replacements:

Bessel function	$\to$	harmonic oscillator eigenfunction
Bessel equation	$\to$	Schrödinger equation for harmonic oscilator

3. Contract the Lie algebra $\mathfrak{su}(2)$ spanned by $J_3, J_\pm$ ($[J_3, J_\pm] = \pm J_\pm$, $[J_+, J_-] = 2J_3$) with respect to the subalgebra J_-. Use a simple Inönü-Wigner contraction to show

$$\begin{array}{lllllll} \lim_{\epsilon\to0} & \epsilon(2J_3) & \to & P & P' & = & \partial_x \\ \lim_{\epsilon\to0} & \epsilon(J_+) & \to & T & T' & = & \partial_t \\ \lim_{\epsilon\to0} & (J_-) & \to & V & V' & = & t\partial_x \end{array}$$

Construct the commutation relations of the contracted operators and show that the operators on the right (P', T', V') satisfy an isomorphic set of commutation relations. The operators $\partial_x,\ \partial_t,\ t\partial_x$ generate the Galilean group in one dimension. Conclude that if the Lie algebra $\mathfrak{a}_1$ is contracted with respect to one of its shift operators the galilean algebra $\mathfrak{gal}(1)$ results.

4. Contract $SO(n+1)$ with respect to the subgroup $SO(n)$ and show how the invariant metric and measure on the sphere $S^n = SO(n+1)/SO(n)$ reduce to the familiar metric and measure on $R^n = ISO(n)/SO(n)$.

5. Disentangling formulas can also be contracted.

a. Use the defining 2×2 matrix representation for $\mathfrak{su}(2)$ to construct the disentangling theorem

$$e^{\zeta J_+ - \zeta^* J_-} = e^{\tau J_+} e^{\log(1+\tau^*\tau) J_3} e^{-\tau^* J_-}$$

and show $\tau = (\zeta/|\zeta|)\tan(|\zeta|)$.

b. Use a faithful matrix representation of the Lie algebra $\mathfrak{h}_4$ to construct the disentangling theorem

$$e^{\alpha a^\dagger - \alpha^* a} = e^{\alpha a^\dagger} e^{-\frac{1}{2}\alpha^*\alpha\ I} e^{-\alpha^* a}$$

c. Use the contraction relation Eq. (13.30) for $\mathfrak{u}(2) \to \mathfrak{h}_4$ to show that the $\mathfrak{u}(2)$ disentangling theorem contracts to the $\mathfrak{h}_4$ disentangling theorem in the limit $\alpha = \lim_{c\to 0} \zeta/c$.

6. Thermal expectation values of the operator X are constructed by taking the trace: $\langle X \rangle = \text{tr}\ X e^{-\beta\mathcal{H}}/\text{tr}\ e^{-\beta\mathcal{H}}$, and a generating function for expectation values is $\langle e^{\alpha X} \rangle = \text{tr}\ \ e^{\alpha X}\ e^{-\beta\mathcal{H}}/\text{tr}\ e^{-\beta\mathcal{H}}$. When the operators X and $\mathcal{H}$ are elements in a finite dimensional Lie algebra these expectation values can often be computed rather simply.

a. Assume $\mathcal{H} = \epsilon J_3$ and X is in the Lie algebra $\mathfrak{su}(2)$. Show that in the 2×2 defining matrix representation

$$e^{\theta\cdot J} \to \begin{bmatrix} \cosh(\theta/2) + (\theta_z/\theta)\sinh(\theta/2) & (\theta_x - i\theta_y)/\theta \sinh(\theta/2) \\ (\theta_x + i\theta_y)/\theta \sinh(\theta/2) & \cosh(\theta/2) - (\theta_z/\theta)\sinh(\theta/2) \end{bmatrix}$$

$$e^{-\beta\mathcal{H}} \to \begin{bmatrix} e^{-\beta\epsilon/2} & 0 \\ 0 & e^{+\beta\epsilon/2} \end{bmatrix}$$

b. Show that the trace of this product is

$$2\cosh(\theta/2)\cosh(\beta\epsilon/2) - 2(\theta_z/\theta)\sinh(\theta/2)\sinh(\beta\epsilon/2) \quad (= 2\cosh(\psi/2))$$

c. Show that in the 2×2 matrix representation with $j = \frac{1}{2}$ and $2j+1 = 2$

$$\langle e^{\theta\cdot J}\rangle = (\sinh\psi/\sinh(\psi/2)) \,/\, (\sinh\beta\epsilon/\sinh(\beta\epsilon/2))$$

d. Show that in the $(2j+1)\times(2j+1)$ dimensional representation

$$\langle e^{\theta\cdot J}\rangle = \frac{\sinh((2j+1)\psi/2)/\sinh(\psi/2)}{\sinh((2j+1)\beta\epsilon/2)/\sinh(\beta\epsilon/2)}$$

e. As j becomes large, show that this ratio simplifies to

$$\langle e^{\theta\cdot J}\rangle \stackrel{j\to\infty}{\longrightarrow} \sinh(j\psi)/\sinh(j\beta\epsilon)$$

f. Contract this generating function to the Heisenberg algebra.

7. One real form of D_3 is the conformal group $SO(4,2)$.

a. Write down the quadratic, cubic, and quartic Casimir operators for $SO(4,2)$. These are analytic continuations of $\mathcal{C}^2 = \sum_{ij} X_{ij}^2$, $\mathcal{C}^3 = \epsilon^{abcdef} X_{ab} X_{cd} X_{ef}$, and $\mathcal{C}^4 = \sum_{ij} Y_{ij}^2$, where $Y_{ij} = \epsilon^{ijcdef} X_{cd} X_{ef}$.

b. Contract $SO(4,2)$ with respect to the subgroup $SO(4)\otimes SO(2)$.

c. Construct the quadratic, cubic, and quartic Casimir operators of the contracted group. These are analytic continuations of the contractions of the three operators of part **a.** If we define $A_i = \lim_{\epsilon\to 0}\epsilon X_{i5}$ and $B_i = \lim_{\epsilon\to 0}\epsilon X_{i6}$, then show that the Casimirs contract to

$$\begin{array}{rcl} \mathcal{C}^2 & \to & \mathbf{A}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} \\ \mathcal{C}^3 & \to & \epsilon^{ijkl} X_{ij} A_k B_l \\ \mathcal{C}^4 & \to & \sum_{ij} (\epsilon^{ijkl} A_k B_l)^2 \end{array}$$

In these expressions the indices range from 1 to 4.

d. Write down the Laplace-Beltrami operators in the eight-dimensional spaces $SO(4,2)/[SO(4)\otimes SO(2)]$ and $I[SO(4)\otimes SO(2)]/[SO(4)\otimes SO(2)]$.

8. Riemannian symmetric spaces have been classified using the Cartan decomposition of simple Lie algebras:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p} \qquad \begin{array}{rcl} [\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h} \\ [\mathfrak{h},\mathfrak{p}] & = & \mathfrak{p} \\ [\mathfrak{p},\mathfrak{p}] & \subseteq & \mathfrak{h} \end{array}$$

Operators X_i span $\mathfrak{h}$ and X_α span $\mathfrak{p}$.

a. Show that the metric on $\mathfrak{p}$ is

$$g_{\alpha,\beta} = C_{\alpha,\gamma}^{\ \ i} C_{\beta,i}^{\ \ \gamma} + C_{\alpha,i}^{\ \ \gamma} C_{\beta,\gamma}^{\ \ i}$$

b. Show that in the contracted limit $Y_\alpha = \lim_{\epsilon \to 0} \epsilon X_\alpha$ a metric tensor on $\mathfrak{p}$ is well defined by

$$g(\mathfrak{p}')_{\alpha,\beta} = \lim_{\epsilon \to 0} (Y_\alpha, Y_\beta)/\epsilon^2 = (X_\alpha, X_\beta)$$

Use the structure constants to show this.

c. Show that this metric is unchanged on the contracted space $P' = G'/H$, as opposed to the metric on $P = G/H$, which varies from place to place on the space.