12

Riemannian Symmetric Spaces

Contents

In the classification of the real forms of the simple Lie algebras we encountered subspaces $\mathfrak{p}$, $i\mathfrak{p}$ on which the Cartan-Killing inner product was negative-definite (on $\mathfrak{p}$) or positive-definite (on $i\mathfrak{p}$). In either case these subspaces exponentiate onto algebraic manifolds on which the invariant metric g_{ij} is definite, either negative or positive. Manifolds with a definite metric are Riemannian spaces. These spaces are also globally symmetric in the sense that every point looks like every other point — because each point in the space $EXP(\mathfrak{p})$ or $EXP(i\mathfrak{p})$ is the image of the origin under some group operation. We briefly discuss the properties of these Riemannian globally symmetric spaces in this chapter.

12.1 Brief Review

In the discussion of the group $SL(2;R)$ we encountered three symmetric spaces. These were $S^2 \sim SU(2)/U(1)$, which is compact, and its dual $H^2_{2+} = SL(2;R)/\ SO(2) = SU(1,1)/U(1)$, which is the upper sheet of the two-sheeted hyperboloid. 'Between' these two spaces occurs $H^2_1 = SL(2;R)/SO(1,1)$, which is the single-sheeted hyperboloid. These spaces are shown in Fig. 12.1.

The Cartan-Killing inner product in the linear vector subspace $\mathfrak{su}(2) - \mathfrak{u}(1)$ is negative definite. This is mapped, under the EXPonential func-

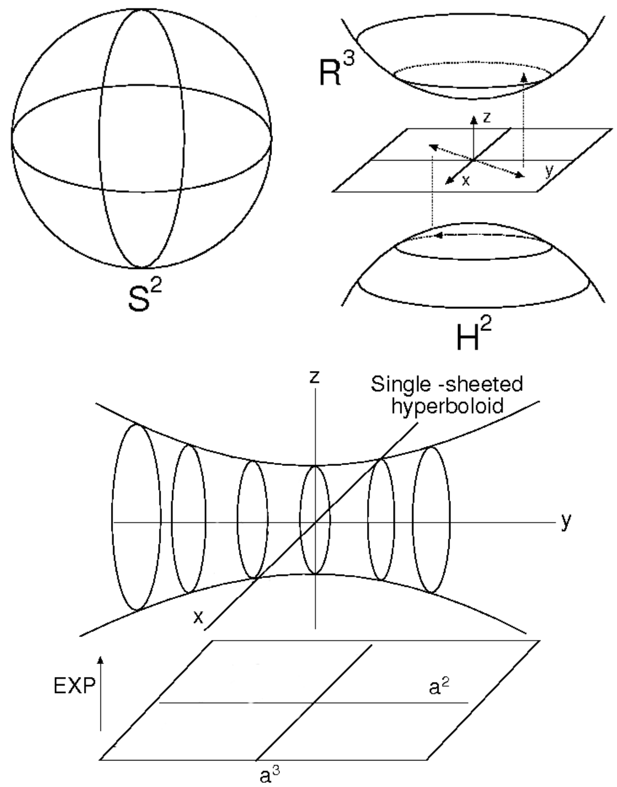

Fig. 12.1. $S^2 = SO(3)/SO(2) = SU(2)/U(1)$, $H^2_{2+} = SO(2,1)/SO(2) = SU(1,1)/U(1)$, $H^2_1 = SO(2,1)/SO(1,1) = SL(2;R)/SO(1,1)$. The first two are Riemannian symmetric spaces, the third is a pseudo-Riemannian symmetric space.

tion, to the Cartan-Killing metric on the space $SU(2)/U(1) \sim S^2$, the sphere. On S^2 the Cartan-Killing metric is negative-definite. We may just as well take it as positive definite. Under this metric the sphere becomes a Riemannian manifold since there is a metric on it with which to measure distances.

The Cartan-Killing inner product on $\mathfrak{su}(1,1) - \mathfrak{u}(1) \simeq \mathfrak{sl}(2;R) - \mathfrak{so}(2)$ is positive definite. It maps to a positive-definite metric on $H^2_{2+} = SU(1,1)/SO(2)$. The upper sheet of the two-sheeted hyperboloid is topologically equivalent to the flat space R^2 but geometrically it is not: it has intrinsic curvature that can be computed, via its Cartan-Killing metric and the curvature tensor derived from it.

The most interesting of these spaces is the single-sheeted hyperboloid H^2_1. It is obtained by exponentiating $\mathfrak{su}(1,1) - \mathfrak{so}(1,1)$. The Cartan-Killing inner product in this linear vector space is indefinite. Therefore the Cartan-Killing metric on the topological space $EXP[\mathfrak{su}(1,1) - \mathfrak{so}(1,1)] = SU(1,1)/SO(1,1)$ is indefinite. The space is a pseudo-Riemannian manifold. In addition it is multiply connected.

12.2 Globally Symmetric Spaces

The three cases for A_1 reviewed in the previous section serve as a model for the description of all other Riemannian symmetric spaces. For a compact simple Lie algebra $\mathfrak{g}$ [i.e. $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, $\mathfrak{sp}(n)$] the Cartan decompositions have the form (11.10)

$$\begin{array}{ccccccccc} \mathfrak{g} & = & \mathfrak{h} & + & \mathfrak{p} & \qquad (\mathfrak{p},\mathfrak{p}) & < & 0 \\ \mathfrak{g}' & = & \mathfrak{h} & + & i\mathfrak{p} & \qquad (i\mathfrak{p},i\mathfrak{p}) & > & 0 \end{array} \tag{12.1}$$

On the linear vector space $\mathfrak{p}$ ($i\mathfrak{p}$) the Cartan-Killing inner product is negative (positive) definite. On the topological spaces $EXP(\mathfrak{p})$ [$EXP(i\mathfrak{p})$] the Cartan-Killing metric is negative (positive) definite also:

$$\begin{array}{cccl} G/H & = & EXP(\mathfrak{p}) & \qquad ds^2 = g_{\mu,\nu}\, dx^\mu dx^\nu < 0 \\ G'/H & = & EXP(i\mathfrak{p}) & \qquad ds^2 = g_{\mu,\nu}\, dx^\mu dx^\nu > 0 \end{array} \tag{12.2}$$

In either case, the metric is definite and defines a Riemannian space. This space is globally symmetric. That is, every point 'looks like' every other point. This is because they all look like the identity $EXP(0)$, since the identity and its neighborhood can be shifted to any other point in the space by multiplication by the appropriate group operation [for example, by $EXP(\mathfrak{p})$ or $EXP(i\mathfrak{p})$].

The space $P = G/H = EXP(\mathfrak{p})$ (e.g. S^2) is compact. The exponential of a straight line through the origin in $\mathfrak{p}$ returns periodically to the neighborhood of the identity. The space P is not topologically equivalent to any Euclidean space, in which a straight line (geodesic) through the origin never returns to the origin. The space P may be simply connected or multiply connected.

The space $P' = G'/H = EXP(i\mathfrak{p})$ (i.e. H^2_{2+}) is noncompact. The exponential of a straight line through the origin in $i\mathfrak{p}$ [a geodesic through the identity in $EXP(i\mathfrak{p})$] simply goes away from this point without ever returning. The space $P' = EXP(i\mathfrak{p})$ is topologically equivalent to a Euclidean space R^n, where $n = \dim i\mathfrak{p}$. Geometrically it is not Euclidean since it has nonzero curvature. This space is simply connected.

The Riemannian spaces $P = EXP(\mathfrak{p})$ and $P' = EXP(i\mathfrak{p})$ are symmetric but not isotropic unless the rank of the space is 1, as it is for S^2 and H^2_{2+}.

If $\mathfrak{g}$ is simple with a Cartan decomposition of the form $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, with standard commutation relations $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, the quotient coset $P = G/K$ is a globally symmetric space as every point "lookslike" every other point.

12.3 Rank

Rank for a symmetric space can be defined in exactly the same way as rank for a Lie group or a Lie algebra. This shouldn't be surprising, as a symmetric space consists of points (coset representatives $P = G/H$ or $P' = G'/H$) in the Lie group.

To compute the rank of a symmetric space one starts from the secular equation for the associated algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$

$$\parallel \mathfrak{Reg}(\mathfrak{h} + \mathfrak{p}) - \lambda I_n \parallel = \sum_{j=0}^{n} (-\lambda)^{n-j} \phi_j(\mathfrak{h}, \mathfrak{p}) \tag{12.3}$$

and restricts to the subspace $\mathfrak{p}$. Calculation of the rank can be carried out in any faithful matrix representation, for example the defining $n \times n$ matrix representation. The secular equations for the spaces $SO(p,q)/SO(p) \times SO(q)$, $SU(p,q)/S[U(p) \times U(q)]$, $Sp(p,q)/Sp(p) \times Sp(q)$ are

$$\left\| \begin{bmatrix} 0 & B \\ B^\dagger & 0 \end{bmatrix} - \lambda I_{p+q} \right\| = \sum_{j=0}^{n=p+q} (-\lambda)^{n-j} \phi_j(B, B^\dagger) \tag{12.4}$$

It is easy to check that the function ϕ_j depends on the $q \times q$ matrix $B^\dagger B$ or the $p \times p$ matrix $BB^\dagger$, whichever is smaller. The rank of these spaces is $\min(p,q)$.

For Riemannian globally symmetric spaces the rank is (cf. Sec. 10.1):

(i) The number of independent functions in the secular equation;
(ii) The number of independent roots of the secular equation;
(iii) The maximal number of mutually commuting operators in the subspace $\mathfrak{p}$ or $\mathfrak{p}'$;
(iv) The number of invariant (Laplace-Beltrami) operators defined over the space P (P');

Table 12.1. *All classical noncompact Riemannian symmetric spaces.*

Root Space	Quotient G'/H	Dimension P	Rank P
A_{p+q-1}	$SU(p,q)/S\,[U(p) \otimes U(q)]$	$2pq$	$\min(p,q)$
A_{n-1}	$SL(n;R)/SO(n)$	$\frac{1}{2}(n+2)(n-1)$	$n-1$
A_{2n-1}	$SU^*(2n)/USp(2n)$	$(2n+1)(n-1)$	$n-1$
B_{p+q}	$SO(p,q)/SO(p) \otimes SO(q)$	pq	$\min(p,q)$
D_{p+q}	$SO(p,q)/SO(p) \otimes SO(q)$	pq	$\min(p,q)$
D_n	$SO^*(2n)/U(n)$	$n(n-1)$	$[n/2]$
C_{p+q}	$USp(2p,2q)/USp(2p) \otimes USp(2q)$	$4pq$	$\min(p,q)$
C_n	$Sp(2n;R)/U(n)$	$n(n+1)$	n

(v) The dimension of a positive-definite root space that can be used to define diagrammatically the properties of these spaces (Araki-Satake root diagrams);

(vi) The number of distinct, nonisotropic directions;

(vii) The dimension of the largest Euclidean submanifold in P.

We will not elaborate on these points here. We mention briefly that the Laplace-Beltrami operators on $P = G/H$ are the Casimir operators of its parent group G, restricted to the subspace P. The number of nonisotropic directions is determined by computing the number of distinct eigenvalues of the Cartan-Killing metric on P, or equivalently and more easily, of the Cartan-Killing inner product on $\mathfrak{p}$ (same as the metric at the identity). In each of the spaces P there is a Euclidean subspace (submanifold). For S^2, any great circle is Euclidean.

12.4 Riemannian Symmetric Spaces

Table 12.1 lists all the classical noncompact Riemannian symmetric spaces of the form G'/H, where G' is simple and noncompact and H is the maximal compact subgroup in G'. To each there is a compact real form under $G'/H \to G/H$. For example, $SO(p,q)/SO(p) \otimes SO(q)$ and $SO(p+q)/SO(p) \otimes SO(q)$ are dual. These spaces are classical because they involve the classical series of Lie groups: the orthogonal, the unitary, and the symplectic.

Table 12.2 lists all the exceptional noncompact Riemannian symmetric spaces. As before, to each there is a dual compact real form.

Table 12.2. *All exceptional noncompact Riemannian symmetric spaces.*

Root Space	G'/H	Dim G'	Dim H	Dim P	Rank P
G_2	$G_{2(+2)}/(A_1 \oplus A_1)$	14	6	8	2
F_4	$F_{4(-20)}/B_4$	52	36	16	1
	$F_{4(+4)}/(C_3 \oplus A_1)$	52	24	28	4
E_6	$E_{6(-26)}/F_4$	78	52	26	2
	$E_{6(-14)}/(D_5 \oplus D_1)$	78	46	32	2
	$E_{6(+2)}/(A_5 \oplus A_1)$	78	38	40	4
	$E_{6(+6)}/C_4$	78	36	42	6
E_7	$E_{7(-25)}/(E_6 \oplus D_1)$	133	79	54	3
	$E_{7(-5)}/(D_6 \oplus A_1)$	133	69	64	4
	$E_{7(+7)}/A_7$	133	63	70	7
E_8	$E_{8(-24)}/(E_7 \oplus A_1)$	248	136	112	4
	$E_{8(+8)}/D_8$	248	120	128	8

12.5 Metric and Measure

The metric tensor on the spaces P, P' is computed by defining a metric at the identity and then moving it elsewhere by group multiplication. The metric at the identity is chosen as the Cartan-Killing inner product on $i\mathfrak{p}$, or its negative on $\mathfrak{p}$.

If $dx(Id)$ are infinitesimal displacements at the Identity that are translated to infinitesimal displacements $dx(p)$ at point p, then these two sets of infinitesimals are linearly related by a nonsingular linear transformation [cf. (4.44)]

$$dx^i(Id) = M^i{}_\mu \, dx^\mu(p) \tag{12.5}$$

The metrics and invariant volume elements are related by [cf. (4.47) and (4.49)]

$$\begin{aligned}
ds^2 &= g_{ij}(Id)dx^i(Id)dx^j(Id) \\
&= g_{\mu\nu}(p)dx^\mu(p)dx^\nu(p) \\
&\Rightarrow \; g_{\mu\nu}(p) = g_{ij}(Id)M^i{}_\mu M^j{}_\nu \\
dV &= \rho(Id)dx^1(Id) \wedge dx^2(Id) \wedge \cdots \wedge dx^n(Id) \\
&= \rho(p)dx^1(p) \wedge dx^2(p) \wedge \cdots \wedge dx^n(p) \\
&\Rightarrow \; \rho(p) = \parallel M(p) \parallel \rho(Id) \sim \sqrt{\det\, g(p)}
\end{aligned} \tag{12.6}$$

The matrix $M^i{}_\mu(p)$ is not easy to compute in general. For the rank-1 spaces $SO(n,1)/SO(n)$, $SU(n,1)/U(n)$, $Sp(n,1)/Sp(n) \times Sp(1)$ defined by

$$P' = \begin{bmatrix} W & X \\ X^\dagger & Y \end{bmatrix} \qquad X = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \qquad (12.7)$$
$$W^2 = I_n + XX^\dagger$$
$$Y^2 = 1 + X^\dagger X$$

the matrix ${M^i}_\mu(X)$ is determined from

$$dx(X) = W dx(Id) \qquad (12.8)$$

The matrix ${M^i}_\mu(X)$ is given by W^{-1}. Since the Cartan-Killing inner product is I_n at the identity, we find

$$\begin{aligned} g_{\mu\nu}(X) &= W^{-1} I_n W^{-1} = \left\{I_n + XX^\dagger\right\}^{-1}_{\mu\nu} \\ \rho(X) &= \parallel W \parallel^{-1} = 1/\sqrt{1 + X^\dagger X} = Y^{-1} \end{aligned} \qquad (12.9)$$

12.6 Applications and Examples

The coset representatives for the Riemannian symmetric spaces $SO(2,1)/SO(2)$ and $SO(3)/SO(2)$ are

$$\begin{array}{cc} SO(2,1)/SO(2) & SO(3)/SO(2) \\ \begin{bmatrix} W & X \\ +X^t & Y \end{bmatrix} & \begin{bmatrix} W & X \\ -X^t & Y \end{bmatrix} \\ W^2 = I_2 + \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} & W^2 = I_2 - \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} \\ Y^2 = I_1 + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & Y^2 = I_1 - \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{array} \qquad (12.10)$$

From these coset representatives we can compute the metric tensors on the noncompact hyperboloid $H_2^2 = SO(2,1)/\ SO(2)$ and the compact sphere $S^2 = SO(3)/SO(2)$. The metric tensors in the two cases are the 2×2 matrices

$$SO(2,1)/SO(2) \qquad\qquad SO(3)/SO(2)$$

$$g_{*,*} = W^{-2} = \left[I_2 + \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} \right]^{-1} \qquad g_{*,*} = W^{-2} = \left[I_2 - \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} \right]^{-1}$$

$$g^{*,*} = W^{+2} = \begin{bmatrix} 1+x^2 & +xy \\ +yx & 1+y^2 \end{bmatrix} \qquad g^{*,*} = W^{+2} = \begin{bmatrix} 1-x^2 & -xy \\ -yx & 1-y^2 \end{bmatrix} \tag{12.11}$$

The noncompact Riemannian symmetric space $H_2^2 = SO(2,1)/SO(2)$ is parameterized by the entire x-y plane while its dual compact Riemannian symmetric space $SO(2+1)/SO(2)$ is parameterized by the interior of the unit circle $Y^2 = 1 - (x^2 + y^2) \geq 0$.

Since the (intrinsic) properties of the Riemannian symmetric space are entirely encoded in its metric tensor, we can begin to compute its important properties, for example, the curvature tensor. It is first useful to compute the Christoffel symbols as a way-station on the road to computing the full Riemannian curvature tensor. The Christoffel symbols (not a tensor!), the Riemannian curvature tensor, the Ricci tensor, and the curvature scalars are constructed in terms of the metric tensor as follows:

$$\begin{array}{lrcl}
\text{Christoffel :} & \Gamma^{\sigma}_{\mu\nu} & = & \dfrac{1}{2} g^{\sigma\alpha} \left(\dfrac{\partial g_{\mu\alpha}}{\partial x^{\nu}} + \dfrac{\partial g_{\nu\alpha}}{\partial x^{\mu}} - \dfrac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right) \\
\text{Riemann C. T. :} & R^{\mu}{}_{\sigma,\alpha\beta} & = & \dfrac{\partial \Gamma^{\mu}_{\sigma\beta}}{\partial x^{\alpha}} - \dfrac{\partial \Gamma^{\mu}_{\sigma\alpha}}{\partial x^{\beta}} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\sigma\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\sigma\alpha} \\
\text{Ricci Tensor :} & R_{\sigma\beta} & = & R^{\mu}{}_{\sigma,\mu\beta} \\
\text{Curvature Scalar :} & R & = & g^{\sigma\beta} R_{\sigma\beta}
\end{array} \tag{12.12}$$

In general, computing these objects is not an easy task. This task is greatly simplified in a symmetric space, for all points look the same and we can compute the tensors wherever the computation is easiest. This turns out to be at the origin. We illustrate by carrying out the computations in the neighborhood of the identity for the compact case, the sphere. Instead of using the pair x, y as coordinates, we use indexed coordinates x^i, $i = 1, 2 \cdots, N$, and set $N = 2$ at the end of this computation.

We first note that it is sufficient to estimate the behavior of the metric tensor in the neighborhood of the origin (identity in the coset) only up to quadratic terms, so that

$$g_{ij} = W^{-2} = \left[I_N - XX^t\right]^{-1}_{ij} \simeq \left[I_N + XX^t\right]_{ij} \rightarrow \delta_{ij} + x^i x^j \qquad (12.13)$$

The inverse (contravariant metric) is $g^{ij} \simeq \delta^{ij} - x^i x^j$, but we will not need this result. In the neighborhood of the identity ($g^{ij} \rightarrow \delta^{ij}$)

$$\begin{aligned} \Gamma^{\sigma}_{\mu\nu} &\rightarrow \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) \\ &= \tfrac{1}{2}\left\{\begin{array}{c} \delta_{\nu\mu}x^{\sigma} + \delta_{\mu\nu}x^{\sigma} - \delta_{\sigma\mu}x^{\nu} \\ \delta_{\nu\sigma}x^{\mu} + \delta_{\mu\sigma}x^{\nu} - \delta_{\sigma\nu}x^{\mu} \end{array}\right\} \qquad (12.14) \\ &\rightarrow \delta_{\mu\nu}x^{\sigma} \qquad (\rightarrow 0 \text{ at origin}) \end{aligned}$$

Computation of the components of the Riemann curvature tensor at the origin is even simpler. At the origin the components of the Christoffel symbols all vanish, so it is sufficient to retain only the first two terms in the expression for the curvature tensor. We find

$$R^{\mu}{}_{\sigma,\alpha\beta} \rightarrow \frac{\partial}{\partial x^{\alpha}}(\delta_{\sigma\beta}x^{\mu}) - \frac{\partial}{\partial x^{\beta}}(\delta_{\sigma\alpha}x^{\mu}) = \delta_{\sigma\beta}\delta_{\alpha}{}^{\mu} - \delta_{\sigma\alpha}\delta_{\beta}{}^{\mu} \qquad (12.15)$$

The contravariant index μ can be lowered with the metric tensor, which is the delta function at the origin, and the resulting fully covariant metric tensor $R_{\mu\sigma,\alpha\beta} = \delta_{\alpha\mu}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\mu}$ exhibits the full spectrum of expected symmetries.

The Ricci tensor is obtained by contraction

$$R_{\sigma\beta} = R^{\mu}{}_{\sigma,\mu\beta} = \delta_{\sigma\beta}\delta_{\mu\mu} - \delta_{\sigma\mu}\delta_{\beta\mu} = N\delta_{\sigma\beta} - \delta_{\sigma\beta} \qquad (12.16)$$

The curvature scalar is obtained from the Ricci tensor by saturating its covariant indices by the contravariant components of the metric tensor, which is simply a delta function at the origin:

$$R = g^{\sigma\beta}R_{\sigma\beta} \rightarrow \delta^{\sigma\beta}(N-1)\delta_{\sigma\beta} = N(N-1) \qquad (12.17)$$

For $N = 2$ (sphere S^2), $R = 2$.

The computation can be carried out just as easily for the noncompact space H_2^2. The major change occurs in the first step, where the metric in the neighborhood of the origin undergoes the change

$$\begin{array}{ccc} SO(2+1)/SO(2) & & SO(2,1)/SO(2) \\ & & \\ g_{ij} \rightarrow \delta_{ij} + x^i x^j & \rightarrow & g_{ij} \rightarrow \delta_{ij} - x^i x^j \end{array} \tag{12.18}$$

The net result is that a negative sign attaches itself at each step in the computation: for example $\Gamma^{\sigma}_{\mu\nu} \rightarrow -\delta_{\mu\nu} x^{\sigma}$. The end result for H_2^2 is that $R = -2$.

12.7 Pseudo Riemannian Symmetic Spaces

Topological spaces on which a 'metric tensor' can be defined that is neither positive definite ($ds^2 = g_{\mu\nu}\ dx^{\mu} dx^{\nu} > 0$, equality $\Rightarrow dx = 0$) nor negative-definite ($ds^2 < 0$), but which is nonsingular ($\parallel g \parallel \neq 0$) are called **pseudo Riemannian spaces**. Pseudo Riemannian spaces that are globally symmetric can be constructed following the procedures described in Sections 12.1 and 12.2. As the example of the single sheeted hyperboloid H_1^2 shows, these spaces are even more interesting than the Riemannian globally symmetric spaces.

To make these statements more explicit, assume a Lie algebra $\mathfrak{g}''$ (noncompact) has a decomposition

$$\mathfrak{g}'' = \mathfrak{h}'' + \mathfrak{p}'' \tag{12.19}$$

with commutation relations of the form (11.10)

$$\begin{array}{rcl} [\mathfrak{h}'', \mathfrak{h}''] & \subseteq & \mathfrak{h}'' \\ [\mathfrak{h}'', \mathfrak{p}''] & \subseteq & \mathfrak{p}'' \\ [\mathfrak{p}'', \mathfrak{p}''] & \subseteq & \mathfrak{h}'' \end{array} \tag{12.20}$$

Then $\mathfrak{h}''$ and $\mathfrak{p}''$ are orthogonal subspaces in $\mathfrak{g}''$ under the Cartan-Killing inner product. Assume also that the inner product is indefinite on $\mathfrak{p}''$ (also $\mathfrak{h}''$). Then

$$P'' = EXP(\mathfrak{p}'') = G''/H'' \tag{12.21}$$

is a pseudo-Riemannian globally symmetric space. The metric on this space is indefinite. The space is curved and typically multiply connected. The space $H'' = EXP(\mathfrak{h}'')$ is also an interesting pseudo Riemannian symmetric space.

All of the algebraic properties associated with a Riemannian symmetric space hold also for pseudo Riemannian symmetric spaces. That is, rank can be defined, and carries most of the implications listed in Section 12.3.

There is a systematic method for constructing pseudo Riemannian symmetric spaces. Begin with a compact simple Lie algebra $\mathfrak{g}$ and suppose T_1, T_2 are two metric-preserving mappings of the Lie algebra onto itself that obey $T_1^2 = I,\ T_2^2 = I$ (cf. Section 11.3) and $T_1 \neq T_2$. Define the eigenspaces of $\mathfrak{g}$ under $T_1,\ T_2$ as $\mathfrak{g}_{\pm,\pm}$:

$$\begin{aligned} T_1\ \mathfrak{g}_{\pm,*} &= \pm\mathfrak{g}_{\pm,*} \\ T_2\ \mathfrak{g}_{*,\pm} &= \pm\mathfrak{g}_{*,\pm} \end{aligned} \tag{12.22}$$

Then T_1 can be used to construct a noncompact algebra

$$\mathfrak{g}' = (\mathfrak{g}_{+,+} + \mathfrak{g}_{+,-}) + i(\mathfrak{g}_{-,+} + \mathfrak{g}_{-,-}) \tag{12.23}$$

and T_2 can be used to split $\mathfrak{g}'$ in a different way

$$\begin{array}{rcccc} \mathfrak{g}'' & = & (\mathfrak{g}_{+,+} + i\mathfrak{g}_{+,-}) & + & (i\mathfrak{g}_{-,+} + \mathfrak{g}_{-,-}) \\ & = & \mathfrak{h}'' & + & \mathfrak{p}'' \end{array} \tag{12.24}$$

The subspaces $\mathfrak{h}'', \mathfrak{p}''$ obey commutation relations (12.20). The Cartan-Killing inner product is indefinite on both $\mathfrak{h}''$ and $\mathfrak{p}''$ as long as $T_1 \neq T_2$.

For $\mathfrak{su}(2)$ the only two mappings are T_1 = block diagonal decomposition and T_2 = complex conjugation. The eigenspace decomposition is

		Operation	$i\sigma_1$	$i\sigma_2$	$i\sigma_3$
T_1	=	Block matrix decomposition	-1	-1	$+1$
T_2	=	Complex conjugation	-1	$+1$	-1
T_3	=	$T_1\ T_2$	$+1$	-1	-1

(12.25)

This gives $\mathfrak{g}_{+,+} = 0,\ \mathfrak{g}_{+,-} = i\sigma_3,\ \mathfrak{g}_{-,+} = i\sigma_2,\ \mathfrak{g}_{-,-} = i\sigma_1$. Note that each mapping T_i has one positive and two negative eigenvalues, and chooses a different generator for the maximal compact subalgebra $\mathfrak{h}'$ of the noncompact real form $\mathfrak{g}'$.

12.8 Conclusion

Globally symmetric spaces have the form $P = G/K$, where $\mathfrak{g}$ is a real form of a simple Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, with $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. All Riemannian globally symmetric spaces are constructed as quotients of a simple Lie group G by a maximal compact subgroup K. More specifically, they are exponentials of a subalgebra $\mathfrak{p}$ of a Lie algebra $\mathfrak{g}$ for which commutation relations and inner products are given by (11.10). Pseudo Riemannian globally symmetric spaces are similarly constructed. For these spaces the rank can be defined. This determines a number of algebraic properties (maximal number of independent mutually commuting generators and Laplace-Beltrami operators) as well as geometric properties (number of nonisotropic directions, dimension of maximal Euclidean subspaces). Metric and measure are determined on these spaces in an invariant way.

12.9 Problems

1. Show that the invariant polynomials $\phi_j(B, B^\dagger)$ in (12.4) actually depend on the invariants of $BB^\dagger$ or $B^\dagger B$. These are the eigenvalues of these square, hermitian matrices. Both the $p \times p$ and $q \times q$ matrix have the same spectrum of nonzero eigenvalues. The remaining $(p-q)$ or $(q-p)$ (whichever is positive) eigenvalues of the larger matrix are zero (Singular Value Decomposition theorem).

2. The second order Laplace-Beltrami operator Δ^2 is constructed from the second order Casimir invariant $\mathcal{C}^2$ by restricting the action of the latter to the Riemannian manifold $G/H = P$.

a. Show that this operator can be expressed in terms of the Cartan-Killing metric tensor on P as $\Delta^2 = g^{ij}(\partial_i \partial_j - \Gamma_{ij}{}^k \partial_k)$.

b. Show that there is one Laplace-Beltrami on the sphere S^2 and compute it in the standard parameterization in terms of the coordinates (x, y) in the interior of the unit disk $x^2 + y^2 \leq 1$.

c. Show that there is one Laplace-Beltrami on the two-sheeted hyperboloid H_2^2 and compute it in the standard parameterization in terms of the coordinates on the plane R^2.

d. Show that these two Laplace-Beltrami operators in dual in some sense. What sense?

e. Extend these results to the sphere S^n and its dual, H^n, $n > 2$.

3. Show that the two metric-preserving mappings T_1 and T_2 that satisfy $T_1^2 = T_2^2 = I$ generate a third, $T_3 = T_1 T_2$ and that $T_1 T_2 = T_2 T_1$. Show that $T_3 \neq I$ if $T_1 \neq T_2$. Show that these three operators, together with the identity, form a group isomorphic with the 'four-group' ('vierergruppe') V_4. Describe the variety of decompositions of a compact

Lie algebra $\mathfrak{g} = \mathfrak{g}_{+,+} + \mathfrak{g}_{+,-} + \mathfrak{g}_{-,+} + \mathfrak{g}_{-,-}$ that is available by choosing first, one of these three involutions, and then a second (there are 3!/1!=6 choices). Discuss dualities.

4. Show that the secular equation for the symmetric space $SO(3)/SO(2)$ can be obtained from (11.2) by setting $b_3 = 0$:

$$\det |\mathfrak{Reg}(\mathfrak{p}) - \lambda I_3| = -\lambda \left[\lambda^2 + (b_1^2 + b_2^2)\right] = 0$$

There is one independent function in this secular equation. There is one independent root. What else can be said about this Riemannian symmetric space?

5. Show that the coefficients $\phi_j(\mathfrak{p})$ in the secular equation for a symmetric space are obtained from the coefficients $\phi_j(\mathfrak{h}, \mathfrak{p})$ in the secular equation for the parent Lie algebra [Eq. (12.3)] by setting $\mathfrak{h} = 0$.

6. The hyperbolic plane H_2^2 is the Riemannian symmetric space $SO(2,1)/SO(2)$ obtained by exponentiating a real symmetric matrix in the three dimensional Lie algebra

$$EXP \left[\begin{array}{c|cc} 0 & t_1 & t_2 \\ \hline t_1 & 0 & 0 \\ t_2 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} x_0 & x_1 & x_2 \\ \hline x_1 & * & * \\ x_2 & * & * \end{array}\right], \qquad x_0^2 - x_1^2 - x_2^2 = 1$$

a. Show that the hyperbolic plane is the two-dimensional algebraic manifold defined by the condition $x_0^2 - x_1^2 - x_2^2 = 1$ in the Lorentz 3-space with signature $(1, 2)$.

b. Show that the invariant metric is induced from the metric $-ds^2 = dx_0^2 - dx_1^2 - dx_2^2$ in this Lorentz 3-space.

c. Use coordinates x_1, x_2 to parameterize the points in H_2^2, and show

$$ds^2 = \frac{\begin{pmatrix} dx_1 & dx_2 \end{pmatrix} \begin{bmatrix} 1 + x_2^2 & -x_1 x_2 \\ -x_1 x_2 & 1 + x_1^2 \end{bmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}}{1 + x_1^2 + x_2^2}$$

d. Show that the invariant measure is

$$d\mu = \frac{dx_1 dx_2}{\sqrt{1 + x_1^2 + x_2^2}}$$

e. Introduce polar coordinates (r, θ), $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$.

Show that

$$ds^2 = \frac{\left(\begin{array}{cc} dr & d\theta \end{array} \right) \left[\begin{array}{cc} \frac{1}{1+r^2} & 0 \\ 0 & r^2 \end{array} \right] \left(\begin{array}{c} dr \\ d\theta \end{array} \right)}{1+r^2}$$

$$d\mu = \frac{r\ dr\ d\theta}{\sqrt{1+r^2}}$$

f. Determine the action of a group operation in $SO(1,2)$ on the point $(x_1, x_2) \in H_2^2$.

7. The metric on a pseudo-Riemannian symmetric space is $g_{ij}(x)$.

a. Show that the generators of infinitesimal rotations at a point are $X_{rs} = g_{rt}x^t\partial_s - g_{st}x^t\partial_r$.

b. Show $[X_{ab}, \Delta] = 0$, where $\Delta = G^{ab;rs}X_{ab}X_{rs}$ is the Laplace-Beltrami operator on this space, $G_{ab;rs} = \text{tr}\ \{\mathfrak{def}(X_{ab})\mathfrak{def}(X_{rs})\}$, and $G^{ab;rs}$ is the inverse of $G_{ab;rs}$.

c. Show that Δ consists of terms that are both quadratic and linear in the operators ∂_r, and that

$$\Delta = g^{rs}\partial_r\partial_s - g^{rs}\Gamma_{rs}{}^t\partial_t$$

The function $\Gamma_{rs}{}^t$ is *not* a tensor. The components of the Christoffel symbol are given by

$$\Gamma_{rs}{}^t = \frac{1}{2}g^{tu}\left(\partial_s g_{ru} + \partial_r g_{su} - \partial_u g_{rs}\right)$$

8. Use radial coordinates $(r, \phi_2, \phi_3, \ldots, \phi_n)$ on the sphere $S^n \subset R^{n+1}$.

a. Show the invariant volume element is

$$dV = \sqrt{\parallel g \parallel} r^{n-1} \sin^{n-2}\phi_2 \sin^{n-3}\phi_3 \ldots \sin^1\phi_{n-1} \sin^0\phi_n\ dr{\wedge}d\phi_2{\wedge}d\phi_3{\wedge}\ldots{\wedge}d\phi_n$$

b. Show that the second order Laplace-Beltrami operator is

$$\Delta = \frac{1}{\sqrt{\parallel g \parallel}}\partial_\mu\sqrt{\parallel g \parallel}g^{\mu\nu}\partial_\nu \qquad \text{where} \qquad \partial_\nu = \partial/\partial\phi_\mu$$

c. Compare this with the second order Casimir operator for $SO(n+1)$:

$$\mathcal{C}_2\left[SO(n+1)/SO(n)\right] = \sum_{1 \leq r < s}^{n+1} X_{r,s}^2(\phi)$$

d. Show that the Laplace-Beltrami operators on a sphere can be written recursively:

$$\Delta(S^n) = \partial_n(f_1(\phi)\partial_n) + f_2(\phi)\Delta(S^{n-1})$$

Compute $f_1(\phi)$ and $f_2(\phi)$.

9. A quantum system with n degrees of freedom is described by a hamiltonian that is a linear superposition of the bilinear products $a_i^\dagger a_j$ ($\mathcal{H} = h_{ij}(t) a_i^\dagger a_j$, $1 \leq i, j \leq n$), so that $i\mathcal{H}$ is a time-dependent element in the Lie algebra $\mathfrak{u}(n)$. Assume the system is initially in its ground state. Show that it evolves into a coherent state whose trajectory exists in the rank one symmetric space $SU(n)/U(n-1)$. Write down the coherent state parameters explicitly for a 2-level system, and relate the coherent state parameters to the forcing terms in the hamiltonian.

10. **Conformal Group:** The inner product on an n-dimensional linear vector space $V^{(n)}$ is defined by $(x,x)_m = m_{ij}x^i x^j$. Define coordinates y in an $n+2$ dimensional linear vector space $W^{(n+2)}$ as follows

$$\begin{array}{lcl} y^i & = & sx^i \quad (1 \leq i \leq n) \\ y^{n+1} & = & s \\ y^{n+2} & = & s(x,x)_m \end{array}$$

and define an inner product M in this space by

$$M = \left[\begin{array}{c|cc} m_{ij} & & \\ \hline & 0 & -\frac{1}{2} \\ & -\frac{1}{2} & 0 \end{array}\right]$$

a. Show $(y,y)_M = M_{\mu\nu}y^\mu y^\nu = (sx,sx)_m - \frac{2}{2}s[s(x,x)_m] = 0$.

b. If m is positive definite and Lie group G preserves inner products in $V^{(n)}$, then $G = O(n)$.

c. Show that the Lie group H that preserves inner products in $W^{(n+2)}$ is $O(n+1,1)$.

d. If the metric m has signature n_1, n_2 $(n_1+n_2=n)$, show that $G = O(n_1,n_2)$ and $H = O(n_1+1, n_2+1)$.

e. H is called a conformal group because it preserves angles. Show this.

f. Construct the quotient space $SO(n_1+1, n_2+1)/SO(n_1,n_2)$.

g. Under a conformal transformation $y \rightarrow y'$ and $x \rightarrow x'$. Show $x'^i = y'^i/y'^{n+1}$.

h. The Lorentz metric $(+1,-1,-1,-1)$ leaves the four-momentum invariant:

$$E^2 - (pc)^2 = (mc^2)^2$$

Show that the conformal group on space time is $SO(4,2)$.

i. Show that the infinitesimal generators of the conformal group are

$$\begin{array}{lcl} L_{\mu\nu} & = & x_\mu \partial_\nu - x_\nu \partial_\mu \\ P_\mu & = & \partial_\mu \\ K_\mu & = & 2x_\mu(x^\nu \partial_\nu) - (x^\nu x_\nu)\partial_\mu = 2x_\mu(x,\partial) - (x,x)\partial_\mu \\ S & = & x^\nu \partial_\nu \end{array}$$

The operators $L_{\mu\nu}$ are the infinitesimal generators of the Lorentz group $SO(3,1)$ and P_μ generate translations. Taken together $L_{\mu\nu}$ and P_μ generate the Poincaré group. The operator S generates dilations and the four operators K_μ generate conformal transformations. Above $x_\mu = g_{\mu\nu}x^\nu$.

j. Show that the additional operators satisfy the commutation relations

$$\begin{array}{rcl} [L_{\mu\nu}, K_\lambda] & = & g_{\nu\lambda}K_\mu - g_{\mu\lambda}K_\nu \\ [L_{\mu\nu}, S] & = & 0 \\ [P_\mu, K_\nu] & = & 2(g_{\mu\nu}S - L_{\mu\nu}) \\ [S, P_\mu] & = & -P_\mu \qquad\qquad [P_\mu, P_\nu] \;=\; 0 \\ [S, K_\mu] & = & +K_\mu \qquad\qquad [K_\mu, K_\nu] \;=\; 0 \end{array}$$

k. Show that $e^{c^\mu K_\mu}(x^\nu) = x'^\nu = \frac{x^\nu + c^\nu(x,x)}{1+2(c,x)+(c,c)(x,x)}$.

l. Show that the conformal group $SO(4,2)$ is:

- The largest group that leaves the free space (no sources) Maxwell Equations form invariant.
- The largest group that maps the (bound, scattering, parabolic) states of the hydrogen atom to themselves.

m. Discuss the duality created by $P_\mu \to P'_\mu = x_\mu$ and $K_\mu \to K'_\mu = 2(x,\partial)\partial_\mu - (\partial,\partial)x_\mu$.

11. The upper half of the complex plane has coordinates $z = x + iy$. This upper half plane provides a well studied model for the hyperbolic plane when a suitable metric is placed on it. The half plane is mapped onto itself by linear fractional transformations

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2;R), \quad ad - bc = 1$$

This transformation group is called the projective special linear transformation group and denoted $PSL(2,Z)$.

a. Show that $M, -M \in SL(2;R)$ generate identical transformations. The group $SL(2;R)$ is a 2-fold covering group of $PSL(2,Z)$.

b. Show

$$z' = \frac{ac(x^2+y^2) + (ad+bc)x + bd \quad + \quad iy}{|cz+d|^2}$$

In particular, show that $y' > 0$ if $y > 0$ and $y' = 0$ if $y = 0$. The transformation maps the upper half plane onto the uper half plane and its boundary, the real axis ($y = 0$), onto itself.

c. Show that the metric

$$ds^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} \frac{1}{y} & 0 \\ 0 & \frac{1}{y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \frac{d\overline{z}dz}{y^2}$$

is invariant under these transformations.

d. Show $dz' = dz/|cz+d|^2$

e. Show that the invariant measure is $d\mu = dxdy/y^2$

f. Show that the distance between two points z_1 and z_2 is

$$s(z_1,z_2) = 2\,\tanh^{-1}\frac{|z_1 - z_2|}{|z_1 - \overline{z}_2|} = \log\left\{\frac{|z_1 - \overline{z}_2| + |z_1 - z_2|}{|z_1 - \overline{z}_2| - |z_1 - z_2|}\right\}$$

12. The unit disk in the complex plane $w = x + iy$ consists of those points that satisfy $\overline{w}w = x^2 + y^2 \leq 1$. The unit disk, with a suitable metric, provides a second representation of the hyperbolic plane. The unit disk is mapped onto itself by linear fractional transformations

$$w \to w' = \frac{\alpha w + \beta}{\overline{\beta} w + \overline{\alpha}}\ , \qquad \left[\begin{array}{cc} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{array} \right] \in SU(1,1), \quad \overline{\alpha}\alpha - \overline{\beta}\beta = 1$$

a. Show that $M, -M \in SU(1,1)$ generate identical mappings of the unit disk into itself.

b. Show that $w = e^{i\phi} \to w' = e^{i\psi}$. Compute $\psi(\phi)$.

c. Show that the metric

$$ds^2 = \left(\begin{array}{cc} dx & dy \end{array} \right) \left[\begin{array}{cc} \frac{1}{(1-\overline{w}w)^2} & 0 \\ 0 & \frac{1}{(1-\overline{w}w)^2} \end{array} \right] \left(\begin{array}{c} dx \\ dy \end{array} \right) = \frac{d\overline{w}dw}{(1-\overline{w}w)^2}$$

is invariant under this group.

d. Show that the invariant volume element is

$$d\mu = \frac{dxdy}{(1-\overline{w}w)^2} = \frac{d\overline{w}dw}{(1-\overline{w}w)^2}$$

e. Show that the distance between two points w_1 and w_2 in this unit disk is

$$s(w_1, w_2) = \tanh^{-1} \left\{ \frac{|w_1 - w_2|}{|1 - w_1\overline{w}_2|} \right\}$$

13. Show that the mapping from z in the upper half plane to w in the unit disk given by

$$w = e^{i\phi} \frac{z - z_0}{z - \overline{z}_0}$$

is conformal, that is, it preserves angles. Here z_0 is any point in the upper half plane.

a. Compute the inverse of this mapping, and show that it maps the interior of the unit disk unto the upper half of the complex plane and the boundary of the unit disk onto the real axis (boundary of the upper half plane).

b. Choose $z_0 = i$ and $e^{i\phi} = i$ to give the canonical map

$$w = \frac{iz + 1}{z + i}$$

c. Show that the matrices that generate the Möbius transformations of the upper half plane and the unit disk are related by

$$S\begin{bmatrix} a & b \\ c & d \end{bmatrix}S^{-1} = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \qquad S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

d. Show that this transformation maps the invariant metric and measure on the upper half plane onto the invariant metric and measure on the unit disk.