11
Real Forms

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Root space diagrams classify all the simple Lie algebras and summarize their commutation relations. The Lie algebras so classified exist over the field of complex numbers. Each simple Lie algebra over C of complex dimension n has a number of inequivalent real subalgebras over R of real dimension n. These are obtained by putting reality restrictions on the coordinates in the complex Lie algebra. The different real forms of a complex simple Lie algebra are obtained systematically by a simple eigenvalue decomposition. For the classical (matrix) Lie algebras, three different procedures suffice to construct all real forms. These are: block submatrix decomposition; subfield restriction; and field embedding.

11.1 Preliminaries

In our attempt to find a canonical form for the commutation relations of a real simple Lie algebra with elements $Z = r^i X_i$ (r^i are real numbers, X_i the generators of the Lie group, or basis vectors in the Lie algebra), we were lead to an eigenvalue equation of the form $\sum\sum r^i[R_i{}^j(Z) - \lambda\delta_i{}^j]X_j = 0$. This equation cannot be solved in general unless the

field is extended from the real to the complex numbers. Allowing that extension, we were able to find a canonical form for the operators in semisimple Lie algebras. The general operator in such algebras has the form

$$\sum_{i=1}^{\text{rank}} h^i H_i + \sum_{\alpha \neq 0} e^\alpha E_\alpha \tag{11.1}$$

where h^i and e^α are complex numbers and the "diagonal" and "shift" operators were defined in Sec. 9.7. The commutation relations were classified in terms of a root space diagram. These diagrams were used to enumerate all the simple Lie algebras over the complex field.

We return now to the question of determining the real forms associated with each of the root space diagrams or, more accurately, the complex Lie algebra associated with each root space diagram. We do this by first presenting Cartan's method of decomposing a Lie algebra into two subspaces with very special commutation relations and orthogonality properties. Three simple decompositions of this type are applied to the compact matrix Lie algebra to generate all the real forms of the classical simple Lie algebras A_{n-1}, D_n, B_n, C_n. These decompositions are: block submatrix decomposition; subfield restriction; and field embeddings.

Example: The noncompact Lie algebras $\mathfrak{sl}(2;R)$ and $\mathfrak{su}(1,1)$ have commutation relations described by the root space A_1. The nonisomorphic Lie algebra $\mathfrak{su}(2)$ has the same root space. To see why, we compute the regular representation of $\mathfrak{sl}(2;R)$ and $\mathfrak{su}(2)$ and their secular equations

Algebra	Defining representation		Regular representation
$\mathfrak{sl}(2;R)$	$\frac{1}{2}\begin{bmatrix} a_1 & a_2+a_3 \\ a_2-a_3 & -a_1 \end{bmatrix}$	$\longrightarrow$	$\begin{bmatrix} 0 & -a_3 & -a_2 \\ a_3 & 0 & a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$
			$-\lambda[\lambda^2 + (-a_1^2 - a_2^2 + a_3^2)] = 0$
$\mathfrak{su}(2)$	$\frac{i}{2}\begin{bmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{bmatrix}$	$\longrightarrow$	$\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$
			$-\lambda[\lambda^2 + (b_1^2 + b_2^2 + b_3^2)] = 0$

(11.2)

In the case of $\mathfrak{sl}(2;R)$ it is possible to find three real roots of the secular equation for certain choices of the real parameters a_1, a_2, a_3 while in the compact case this is not possible. If the real parameters (a_1, a_2, a_3) and (b_1, b_2, b_3) are allowed to become complex the two Lie algebras become algebras of 2×2 complex traceless matrices — the Lie algebra for $SL(2;C)$. This relation is shown in Fig. 11.

The complex extension Lie algebra has root space A_1 describing canonical commutation relations for the diagonal and shift operators shown in Fig. 11.1. The most general element in this Lie algebra is a complex linear combination of the three matrices shown. The algebras $\mathfrak{sl}(2;R)$ and $\mathfrak{su}(2)$ have real dimension 3 while their common complex extension has complex dimension 3 (= real dimension 6). In the following sections we present a systematic way for determining how to restrict the complex parameters to real parameters in order to construct all inequivalent real Lie algebras with the same dimension as the complex Lie algebra whose commutation relations are described by a root space diagram.

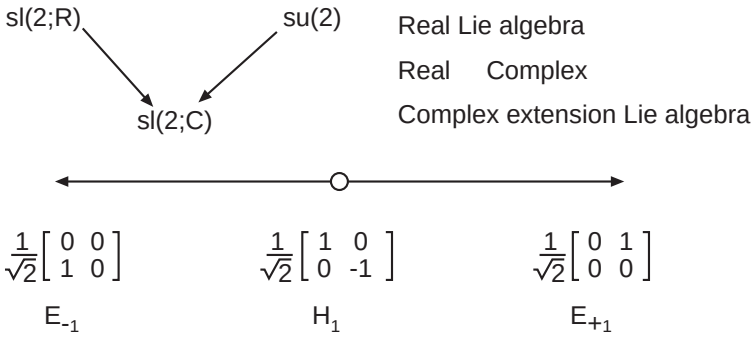

Fig. 11.1. Lie groups $SL(2;R)$ and $SU(2)$ are related by analytic continuation. The canonical form for the diagonal and shift operators in their Lie algebras is also shown.

11.2 Compact and Least Compact Real Forms

The Cartan-Killing inner product for the basis vectors H_i, E_α is

$$\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & & & & \\ & & & & & 0 & 1 & & \\ & & & & & 1 & 0 & & \\ & & & & & & & 0 & 1 \\ & & & & & & & 1 & 0 \\ & & & & & & & & & \ddots \end{bmatrix} \begin{matrix} H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_n \\ E_\alpha \\ E_{-\alpha} \\ E_\beta \\ E_{-\beta} \\ \vdots \end{matrix} \tag{11.3}$$

The inner product can be brought to diagonal form by choosing linear combinations of basis vectors $\frac{1}{\sqrt{2}}(E_\alpha \pm E_{-\alpha})$:

$$\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & -1 & & \\ & & & & & & & 1 & \\ & & & & & & & & -1 \\ & & & & & & & & & \ddots \end{bmatrix} \begin{matrix} H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_n \\ \frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}) \\ \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}) \\ \frac{1}{\sqrt{2}}(E_\beta + E_{-\beta}) \\ \frac{1}{\sqrt{2}}(E_\beta - E_{-\beta}) \\ \vdots \end{matrix} \tag{11.4}$$

If we restrict the coefficients of H_i, $\frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha})$, and $\frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha})$ (all $\alpha \neq 0$) to be real, then the generators $\frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha})$ span the maximal compact subalgebra (closure under commutation must be verified — this is left an an exercise) while the generators H_i and $\frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha})$ span a noncompact subspace.

On the other hand, if we restrict the coefficients of H_i and $\frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha})$ to be imaginary and those of $\frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha})$ to be real

$$ih^i H_i + ie^\alpha \frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}) + e^{-\alpha} \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}) \tag{11.5}$$

then the factors i can be absorbed within the generators. With respect to

these redefined generators the Cartan-Killing inner product is negative definite and the algebra constructed is compact

$$\begin{array}{c} h^1 \\ h^2 \\ h^3 \\ \vdots \\ h^n \\ e^{+\alpha} \\ e^{-\alpha} \\ e^{+\beta} \\ e^{-\beta} \\ \vdots \end{array} \begin{bmatrix} -1 & & & & & & & & & \\ & -1 & & & & & & & & \\ & & -1 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & -1 & & & & & \\ & & & & & -1 & & & & \\ & & & & & & -1 & & & \\ & & & & & & & -1 & & \\ & & & & & & & & -1 & \\ & & & & & & & & & \ddots \end{bmatrix} \begin{array}{c} iH_1 \\ iH_2 \\ iH_3 \\ \vdots \\ iH_n \\ i\frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}) \\ \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}) \\ i\frac{1}{\sqrt{2}}(E_\beta + E_{-\beta}) \\ \frac{1}{\sqrt{2}}(E_\beta - E_{-\beta}) \\ \vdots \end{array} \tag{11.6}$$

real coefficients — Cartan Killing inner product — basis vectors in Lie algebra

Two real forms of A_1, $\mathfrak{sl}(2;R)$ and $\mathfrak{su}(1,1)$, are obtained as follows

$$\begin{array}{ccc} H_1 & \frac{1}{\sqrt{2}}(E_{+1} + E_{-1}) & \frac{1}{\sqrt{2}}(E_{+1} - E_{-1}) \end{array}$$

$$(h_r + ih_i)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (a_r + ia_i)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (b_r + ib_i)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{11.7}$$

$$(h_r, a_r, b_r) \longrightarrow \begin{bmatrix} h_r & a_r + b_r \\ a_r - b_r & -h_r \end{bmatrix} \qquad \mathfrak{sl}(2;R) \tag{11.8}$$

$$(ih_i, ia_i, b_r) \longrightarrow i\begin{bmatrix} h_i & a_i - ib_r \\ a_i + ib_r & -h_i \end{bmatrix} \qquad \mathfrak{su}(2)$$

Here the six parameters $h_r, h_i;\ a_r, a_i;\ b_r, b_i$ are real.

It is useful to specify how compact a real form is by specifying its **index** (n_+, n_-), where n_+ is the dimension of the subspace on which the nonsingular Cartan-Killing inner product is positive definite and n_- is the dimension of the subspace (subalgebra) on which it is negative definite. These two pieces of information may be abbreviated to a single integer, the **character** $\chi = n_+ - n_-$, to describe a real form. This is the trace of the normalized Cartan-Killing form. Inspection of (11.3) and (11.6) shows that the character is +(rank) of the root space for

the real Lie algebra spanned by H_i, $\frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha})$, and $\frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha})$ and is $-$(dimension) for the compact real form spanned by real linear combinations of $iH_i, i\frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha})$, and $\frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha})$. In general, for all real forms the character satisfies the bounds

$$-\text{ dimension} \le \chi = \text{ character } \le +\text{rank} \tag{11.9}$$

11.3 Cartan's Procedure for Constructing Real Forms

Cartan has proposed a simple and elegant procedure for constructing all the real forms of a (complex) simple Lie algebra. This procedure constructs one real form from another by "analytic continuation." It is modeled on Minkowski's transformation of spacetime (x, y, z, ct) with indefinite metric $g_{\mu,\nu} = \text{diag}(1, 1, 1, -1)$ to spacetime with imaginary time (x, y, z, ict) and positive-definite metric $g_{\mu,\nu} = \text{diag}(1, 1, 1, 1)$.

Since the compact real form can always be constructed easily for a simple Lie algebra [cf. (11.6)] it is useful to begin with that form. The compact Lie algebra $\mathfrak{g}$ is divided into two pieces with the following commutation relations and orthogonality properties

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$$

$$\begin{array}{lcllcl} [\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h} & (\mathfrak{h},\mathfrak{h}) & < & 0 \\ [\mathfrak{h},\mathfrak{p}] & \subseteq & \mathfrak{p} & (\mathfrak{h},\mathfrak{p}) & = & 0 \\ [\mathfrak{p},\mathfrak{p}] & \subseteq & \mathfrak{h} & (\mathfrak{p},\mathfrak{p}) & < & 0 \end{array} \tag{11.10}$$

In short, the subspace $\mathfrak{h}$ is a subalgebra and $\mathfrak{p}$ is its orthogonal complement. A concrete example of this decomposition is

$$\begin{array}{ccccc} \mathfrak{su}(2) & = & \mathfrak{u}(1) & + & \mathfrak{su}(2) - \mathfrak{u}(1) \end{array}$$

$$\frac{i}{2}\begin{bmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{bmatrix} = \frac{i}{2}\begin{bmatrix} a_3 & 0 \\ 0 & -a_3 \end{bmatrix} + \frac{i}{2}\begin{bmatrix} 0 & a_1 - ia_2 \\ a_1 + ia_2 & 0 \end{bmatrix} \tag{11.11}$$

The Lie algebra $\mathfrak{g}$ is mapped into a noncompact Lie algebra $\mathfrak{g}'$ by means of "Minkowski's trick": $\mathfrak{p} \to \mathfrak{p}' = i\mathfrak{p}$. The mapping, commutation relations, and orthogonality relations are

$$\begin{array}{ccccccc} \mathfrak{g} & = & \mathfrak{h} + \mathfrak{p} & \longrightarrow & \mathfrak{g}' & = & \mathfrak{h} + i\mathfrak{p} = \mathfrak{h} + \mathfrak{p}' \end{array}$$

$$\begin{array}{lcllcl} [\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h} & (\mathfrak{h},\mathfrak{h}) & < & 0 \\ [\mathfrak{h},\mathfrak{p}'] & \subseteq & \mathfrak{p}' & (\mathfrak{h},\mathfrak{p}') & = & 0 \\ [\mathfrak{p}',\mathfrak{p}'] & \subseteq & \mathfrak{h} & (\mathfrak{p}',\mathfrak{p}') & > & 0 \end{array} \tag{11.12}$$

In $\mathfrak{g}'$, $\mathfrak{h}$ is the maximal compact subalgebra and $\mathfrak{p}'$ consists of all the noncompact generators. The character of this algebra is

$$\chi(\mathfrak{g}') = \dim(\mathfrak{p}') - \dim(\mathfrak{h}) = \dim(\mathfrak{g}) - 2\dim(\mathfrak{h}) = 2\dim(\mathfrak{p}') - \dim(\mathfrak{g}) \tag{11.13}$$

As a concrete example of this mapping, we have from (11.11)

$$\mathfrak{su}(2) \rightarrow \mathfrak{su}(1,1): \qquad \frac{i}{2}\begin{bmatrix} a_3 & 0 \\ 0 & -a_3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & a_1 - ia_2 \\ a_1 + ia_2 & 0 \end{bmatrix} \tag{11.14}$$

The mapping is reversible: noncompact $\mathfrak{g}'$ can be mapped back to compact $\mathfrak{g}$.

A systematic way exists for finding Cartan decompositions (11.12). Assume T is a linear mapping of the Lie algebra $\mathfrak{g}$ onto itself that preserves inner products, and that also obeys

$$T^2 = I \tag{11.15}$$

("involutive automorphism"). Then T has two eigenvalues: ± 1. Under T, one eigenspace of T is mapped into itself while the other (its orthogonal complement) is mapped into its negative. The map T splits $\mathfrak{g}$ into eigenspaces $\mathfrak{h}$ and $\mathfrak{p}$

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{h} & + & \mathfrak{p} \\ T(\mathfrak{g}) & = & T(\mathfrak{h}) & + & T(\mathfrak{p}) \\ \mathfrak{g} & = & (+1)\mathfrak{h} & + & (-1)\mathfrak{p} \end{array} \tag{11.16}$$

The two subspaces are orthogonal

$$(\mathfrak{h}, \mathfrak{p}) = (T^2\mathfrak{h}, \mathfrak{p}) = (T\mathfrak{h}, T\mathfrak{p}) = (\mathfrak{h}, -\mathfrak{p}) = -(\mathfrak{h}, \mathfrak{p}) = 0 \tag{11.17}$$

and satisfy commutation relations (11.12).

As a consequence of this result, a search for all real forms of a complex semisimple Lie algebra reduces to a hunt for all metric-preserving mappings T of the compact real form of that Lie algebra to itself that obey $T^2 = I$.

11.4 Real Forms of Simple Matrix Lie Algebras

All of the real forms of all of the simple classical (matrix) Lie algebras can be constructed from one of three types of mappings T of matrices into themselves that obey $T^2 = I$. These three mapping types are derived from block matrix decomposition, subfield restriction, and field embeddings. We discuss each in the next three subsections, indicating

the real forms that are produced. In all instances we begin with the compact Lie algebras.

11.4.1 Block Matrix Decomposition

In a block matrix decomposition the compact Lie algebras $\mathfrak{u}(n, F)$ have the form

$$\mathfrak{u}(n; F): \qquad \begin{bmatrix} A_p & 0 \\ 0 & A_q \end{bmatrix} + \begin{bmatrix} 0 & B \\ -B^\dagger & 0 \end{bmatrix} \tag{11.18}$$

where $A_p = -A_p^\dagger$, $A_q = -A_q^\dagger$, and B is an arbitrary $p \times q$ matrix. Under the procedure described in the previous section the off-diagonal block is multiplied by i. This is equivalent to changing the metric I_{p+q} that is preserved by $\mathfrak{u}(n; F)$ to the metric $I_{p,q}$ that is preserved by $\mathfrak{u}(p, q; F)$, where $p+q = n$. The factor i can be absorbed into the $p \times q$ off-diagonal blocks, so that the noncompact algebra has matrix form

$$\mathfrak{u}(p, q; F): \qquad \begin{bmatrix} A_p & 0 \\ 0 & A_q \end{bmatrix} + \begin{bmatrix} 0 & B \\ +B^\dagger & 0 \end{bmatrix} \tag{11.19}$$

For the fields $F = R, C, Q$ related to the root spaces $(D, B), A, C$ the real forms are

$$\begin{array}{lll} R: & \mathfrak{so}(p, q) & D, B \\ C: & \mathfrak{su}(p, q) & A \\ Q: & \mathfrak{sp}(p, q) & C \end{array} \tag{11.20}$$

11.4.2 Subfield Restriction

The real numbers form a subset (subfield) of the complex numbers; the complex numbers form a subset (subfield) of the quaternions. A Lie algebra over the complex numbers can be divided into two subsets: real matrices and the remainder, imaginary matrices. Similarly, a matrix algebra over the quaternions can be divided into two subsets: complex matrices and the remainder:

$$\begin{array}{ccccccc} \mathfrak{g} & = & \mathfrak{h} & + & \mathfrak{p} & \longrightarrow & \mathfrak{g}' \\ \mathfrak{su}(n) & = & \mathfrak{so}(n) & + & [\mathfrak{su}(n) - \mathfrak{so}(n)] & \longrightarrow & \mathfrak{sl}(n; R) \\ \mathfrak{sp}(n) & = & \mathfrak{u}(n) & + & [\mathfrak{sp}(n) - \mathfrak{u}(n)] & \longrightarrow & \mathfrak{sp}(2n; R) \end{array} \tag{11.21}$$

Under the Cartan procedure, $\mathfrak{su}(n)$ is mapped to $\mathfrak{sl}(n; R)$ and $\mathfrak{sp}(n)$ is mapped to $\mathfrak{sp}(2n; R)$.

We illustrate this for $\mathfrak{su}(2)$:

$$\begin{array}{ccccccc}
\mathfrak{su}(2) = \frac{1}{2}\begin{bmatrix} ia_3 & ia_1+a_2 \\ ia_1-a_2 & -ia_3 \end{bmatrix} & \to & \frac{1}{2}\begin{bmatrix} 0 & +a_2 \\ -a_2 & 0 \end{bmatrix} & + & \frac{i}{2}\begin{bmatrix} a_3 & a_1 \\ a_1 & -a_3 \end{bmatrix} \\
 & & \downarrow & & \mathfrak{p} \to i\mathfrak{p} \downarrow \\
\mathfrak{sl}(2;R) = \frac{1}{2}\begin{bmatrix} a_3 & a_1+a_2 \\ a_1-a_2 & -a_3 \end{bmatrix} & \leftarrow & \frac{1}{2}\begin{bmatrix} 0 & +a_2 \\ -a_2 & 0 \end{bmatrix} & + & \frac{1}{2}\begin{bmatrix} a_3 & a_1 \\ a_1 & -a_3 \end{bmatrix}
\end{array} \tag{11.22}$$

The transformation from $\mathfrak{sp}(n) = \mathfrak{u}(n;Q)$ to $\mathfrak{sp}(2n;R)$ is somewhat less familiar. To make the mapping more comprehensible, it is useful to recall the mappings of complex numbers into real 2×2 matrices and of quaternions into complex 2×2 matrices [cf. (3.3) and (3.4)]

$$\begin{array}{rcl}
\alpha + i\beta & \longrightarrow & \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \\
\alpha + \mathcal{I}\beta + \mathcal{J}\gamma + \mathcal{K}\delta & \longrightarrow & \begin{bmatrix} \alpha+i\delta & i\beta+\gamma \\ i\beta-\gamma & \alpha-i\delta \end{bmatrix}
\end{array} \tag{11.23}$$

where α, β, γ and δ are real. With these replacements the Lie algebra of $n\times n$ complex matrices for $\mathfrak{u}(n)$ is replaced by a set of $2n\times 2n$ real matrices. We call these matrices $\mathfrak{ou}(2n)$, since they form an orthogonal representation of the unitary algebra in terms of $2n\times 2n$ matrices. Similarly, the Lie algebra of $n\times n$ quaternion matrices for $\mathfrak{sp}(n)$ is replaced by a set of $2n\times 2n$ complex matrices $\mathfrak{usp}(2n)$, the unitary representation of the symplectic algebra of $2n\times 2n$ matrices:

$$\begin{array}{ccc}
 & \overset{(11.23)}{} & \\
\mathfrak{u}(n) & \longrightarrow & \mathfrak{ou}(2n) \\
\mathfrak{sp}(n) & \longrightarrow & \mathfrak{usp}(2n)
\end{array} \tag{11.24}$$

Since $\mathfrak{usp}(2n)$ consists of complex matrices, the algebra can be decomposed into the subalgebra of real matrices, which is $\mathfrak{ou}(2n)$, and the

complementary subspace of imaginary matrices

$$\begin{array}{ccccc}
\mathfrak{sp}(n) & = & \mathfrak{u}(n) & + & [\mathfrak{sp}(n) - \mathfrak{u}(n)] \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{usp}(2n) & = & \underbrace{\mathfrak{ou}(2n)}_{\text{real}} & + & \underbrace{[\mathfrak{usp}(2n) - \mathfrak{ou}(2n)]}_{\text{imaginary}} \\
\downarrow & & \downarrow & & \downarrow \mathfrak{p} \to i\mathfrak{p} \\
\mathfrak{sp}(2n;R) & = & \underbrace{\mathfrak{ou}(2n)}_{\text{real}} & + & \underbrace{i\,[\mathfrak{usp}(2n) - \mathfrak{ou}(2n)]}_{\text{real}}
\end{array} \tag{11.25}$$

Both $\mathfrak{sl}(n;R)$ and $\mathfrak{sp}(2n;R)$ are the least compact real forms associated with their respective root spaces A_{n-1} and C_n.

Remark: The matrix Lie group $Sp(2n;R)$ leaves invariant a nonsingular antisymmetric metric in R^{2n}. It is possible to choose coordinates $p_1, q_1,\ p_2, q_2, \ldots,\ p_n, q_n$ in this space so that the inner product between two vectors $v_i^{'} G_{ij} v_j$ is

$$v_i^{'} G_{ij} v_j = (p_1, q_1,\ p_2, q_2, \ldots,\ p_n, q_n)' \left[\begin{array}{cc|cc|c|cc} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ \hline & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ \hline & & & & \ddots & & \\ & & & & & \ddots & \\ \hline & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{array}\right] \left[\begin{array}{c} p_1 \\ q_1 \\ p_2 \\ q_2 \\ \vdots \\ \vdots \\ p_n \\ q_n \end{array}\right]$$

$$= \sum_{i=1}^{n} (p_i' q_i - q_i' p_i) \tag{11.26}$$

Then symplectic transformations $M \in Sp(2n;R)$ leave this metric matrix G invariant: $M^t G M = G$. Symplectic transformations leave invariant the canonical form of the Hamiltonian equations of motion in Classical Mechanics.

Table 11.1. *Real forms of the simple classical Lie algebras.*

Mapping	Real Form	Root Space	Condition
Block Submatrix	$\mathfrak{so}(p,q)$	D_n	$p+q=2n$
	$\mathfrak{so}(p,q)$	B_n	$p+q=2n+1$
	$\mathfrak{su}(p,q)$	A_{n-1}	$p+q=n$
	$\mathfrak{sp}(p,q)$	C_n	$p+q=n$
Subfield Restriction	$\mathfrak{sl}(n;R)$	A_{n-1}	
	$\mathfrak{sp}(2n;R)$	C_n	
Field Embedding	$\mathfrak{so}^*(2n)$	D_n	
	$\mathfrak{su}^*(2n)$	A_{2n-1}	

11.4.3 Field Embeddings

The algebras for the orthogonal and unitary groups of even-dimension have the following decompositions:

$$\begin{array}{ccccccc}
\mathfrak{so}(2n) & = & \mathfrak{ou}(2n) & + & [\mathfrak{so}(2n)-\mathfrak{ou}(2n)] & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathfrak{ou}(2n) & + & i\,[\mathfrak{so}(2n)-\mathfrak{ou}(2n)] & = & \mathfrak{so}^*(2n) \\
 & & & & & & \\
\mathfrak{su}(2n) & = & \mathfrak{usp}(2n) & + & [\mathfrak{su}(2n)-\mathfrak{usp}(2n)] & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathfrak{usp}(2n) & + & i\,[\mathfrak{su}(2n)-\mathfrak{usp}(2n)] & = & \mathfrak{su}^*(2n)
\end{array} \tag{11.27}$$

Application of the map $\mathfrak{p} \to i\mathfrak{p}$ produces the real forms $\mathfrak{so}^*(2n)$ and $\mathfrak{su}^*(2n)$.

Remark: The real forms $\mathfrak{so}^*(2n)$ of D_n and $\mathfrak{su}^*(2n)$ of A_{2n-1} do not explicitly occur in the list of matrix Lie algebras given in Chapter 5.

11.5 Results

We summarize in Table 11.1 the real forms of the simple classical Lie algebras. This table indicates the root space associated with each real form.

Some of the low-dimensional root spaces are equivalent. For example, A_1 (where the compact real form is $\mathfrak{su}(2)$), B_1 ($\mathfrak{so}(3)$), and C_1 ($\mathfrak{sp}(1)$) are equivalent, as are B_2 ($\mathfrak{so}(5)$) and C_2 ($\mathfrak{sp}(2)$). So also are A_3 ($\mathfrak{su}(4)$) and D_3 ($\mathfrak{so}(6)$). As a result, there are equivalences between the real forms of these Lie algebras. These equivalences are summarized in Table 11.2.

For completeness, we list the real forms for the exceptional Lie alge-

Table 11.2. *Equivalence among real forms of the simple classical Lie algebras.*

A_1	$\sim$	B_1	$\sim$	C_1
$\mathfrak{su}(2)$	$\sim$	$\mathfrak{so}(3)$	$\sim$	$\mathfrak{sp}(1) = \mathfrak{usp}(2)$
$\mathfrak{su}(1,1) = \mathfrak{sl}(2;R)$	$\sim$	$\mathfrak{so}(2,1)$	$\sim$	$\mathfrak{sp}(2;R)$
D_2	$=$	A_1	$+$	A_1
$\mathfrak{so}(4)$	$=$	$\mathfrak{so}(3)$	$+$	$\mathfrak{so}(3)$
$\mathfrak{so}^*(4)$	$\sim$	$\mathfrak{so}(3)$	$+$	$\mathfrak{so}(2,1)$
$\mathfrak{so}(3,1)$	$\sim$	$\mathfrak{sl}(2;C)$		
$\mathfrak{so}(2,2)$	$\sim$	$\mathfrak{so}(2,1)$	$+$	$\mathfrak{so}(2,1)$
B_2	$=$	C_2		
$\mathfrak{so}(5)$	$\sim$	$\mathfrak{sp}(2) = \mathfrak{usp}(4)$		
$\mathfrak{so}(4,1)$	$\sim$	$\mathfrak{sp}(1,1) = \mathfrak{usp}(2,2)$		
$\mathfrak{so}(3,2)$	$\sim$	$\mathfrak{sp}(4;R)$		
D_3	$=$	A_3		
$\mathfrak{so}(6)$	$\sim$	$\mathfrak{su}(4)$		
$\mathfrak{so}(5,1)$	$\sim$	$\mathfrak{su}^*(4)$		
$\mathfrak{so}^*(6)$	$\sim$	$\mathfrak{su}(3,1)$		
$\mathfrak{so}(4,2)$	$\sim$	$\mathfrak{su}(2,2)$		
$\mathfrak{so}(3,3)$	$\sim$	$\mathfrak{sl}(4;R)$		

bras in Table 11.3. The subscript in parenthesis after the rank is the character of the real form.

11.6 Conclusion

Connected root space diagrams summarize the commutation relations of semisimple Lie algebras over the field of complex numbers. By placing various reality restrictions on the coefficients of the complex algebra, a spectrum of real subalgebras is obtained, each of which has the same complex extension. To each root space there corresponds a unique real form that is compact. All other real forms are obtained from this compact real form by 'analytic continuation.' The analytic continuation is carried out by determining all linear mappings T on the compact algebra $\mathfrak{g}$ that preserve the inner product and obey $T^2 = I$. The subspace $\mathfrak{p}$ of $\mathfrak{g}$ that obeys $T(\mathfrak{p}) = -\mathfrak{p}$ is analytically continued by $\mathfrak{p} \to \mathfrak{p}' = i\mathfrak{p}$; the subspace $\mathfrak{h}$ of $\mathfrak{g}$ that obeys $T(\mathfrak{h}) = +\mathfrak{h}$ is the maximal compact subalgebra of the noncompact real form $\mathfrak{g}'$: $\mathfrak{g} = \mathfrak{h} + \mathfrak{p} \to \mathfrak{g}' = \mathfrak{h} + i\mathfrak{p}'$. For the simple classical (matrix) Lie algebras three types of mappings T suffice

Table 11.3. *Real forms of the exceptional Lie algebras.*

Root Space	$\text{Class}_{\text{Rank(Character)}}$	Maximal Compact Subgroup Root Space	Dimension
G_2	$G_{2(-14)}$	G_2	14
	$G_{2(+2)}$	$A_1 + A_1$	6
F_4	$F_{4(-52)}$	F_4	52
	$F_{4(-20)}$	B_4	36
	$F_{4(+4)}$	$C_3 + A_1$	24
E_6	$E_{6(-78)}$	E_6	78
	$E_{6(-26)}$	F_4	52
	$E_{6(-14)}$	$D_5 + D_1$	46
	$E_{6(+2)}$	$A_5 + A_1$	38
	$E_{6(+6)}$	C_4	36
E_7	$E_{7(-133)}$	E_7	133
	$E_{7(-25)}$	$E_6 + D_1$	79
	$E_{7(-5)}$	$D_6 + A_1$	69
	$E_{7(+7)}$	A_7	63
E_8	$E_{8(-248)}$	E_8	248
	$E_{8(-24)}$	$E_7 + A_1$	136
	$E_{8(+8)}$	D_8	120

to construct all real forms. These are: block submatrix decomposition; subfield restriction; and field embedding.

11.7 Problems

1. Four operators $a_i^\dagger a_j$ can be constructed from boson operators for two modes $1 \leq i, j \leq 2$. These operators close under commutation.

a. Show that the regular representation of this Lie algebra is

$$\mathfrak{Reg}(wa_1^\dagger a_1 + xa_1^\dagger a_2 + ya_2^\dagger a_1 + za_2^\dagger a_2) = \begin{bmatrix} 0 & -x & y & 0 \\ -y & w-z & 0 & y \\ x & 0 & -w+z & -x \\ 0 & x & -y & 0 \end{bmatrix}$$

b. Show that the Cartan-Killing inner product is

$$\text{tr } \mathfrak{Reg}^2(wa_1^\dagger a_1 + xa_1^\dagger a_2 + ya_2^\dagger a_1 + za_2^\dagger a_2) = 2(w-z)^2 + 8xy$$

c. Set

$$\begin{array}{rclcrcl} w & = & \alpha+\beta & & x & = & \gamma+\delta \\ z & = & \alpha-\beta & & y & = & \gamma-\delta \end{array} \qquad \text{inner product} \quad \to 8(\beta^2+\gamma^2-\delta^2)$$

c. $wa_1^\dagger a_1 + xa_1^\dagger a_2 + ya_2^\dagger a_1 + za_2^\dagger a_2 = \alpha(a_1^\dagger a_1 + a_2^\dagger a_2) + \beta(a_1^\dagger a_1 - a_2^\dagger a_2) + \gamma(a_1^\dagger a_2 + a_2^\dagger a_1) + \delta(a_1^\dagger a_2 - a_2^\dagger a_1)$. Conclude: $a_1^\dagger a_1 + a_2^\dagger a_2$ spans the maximum commutative subalgebra, $a_1^\dagger a_2 - a_2^\dagger a_1$ spans the maximal compact subalgebra, and the two generators $a_1^\dagger a_1 - a_2^\dagger a_2$ and $a_1^\dagger a_2 + a_2^\dagger a_1$ are noncompact.

d. Identify the simple three dimensional subalgebra as $\mathfrak{sl}(2;R)$ or $\mathfrak{su}(1,1)$. Show that the compact real form is obtained by multiplying the two noncompact generators by i.

e. Construct a 2×2 matrix representation of the three operators that span $\mathfrak{su}(1,1)$ using the methods of Chapter 6. Multiply the two noncompact operators by i. Show that the three matrices that result are exactly $i\sigma_j$, where σ_j are the Pauli spin matrices.

2. The classical matrix groups $SO(n)$ are not simply connected, so they are $k \to 1$ images of their universal covering groups $\overline{SO(n)} = Spin(n)$, for some integer k. Show that the covering groups $Spin(n)$ are classical matrix groups for $n = 3, 4, 5, 6$, and make these identifications:

$$\begin{array}{rcl} Spin(3) & = & SU(2) \\ Spin(4) & = & SU(2)\otimes SU(2) \\ Spin(5) & = & USp(4) \\ Spin(6) & = & SU(4) \end{array}$$

Show that for $n > 6$ the groups $Spin(n)$ are not equal to any classical matrix Lie groups.

3. **Spectrum of Quadratic Casimir:**

a. Use the metric (11.3) for a simple Lie algebra to show that the quadratic Casimir operator is

$$\mathcal{C}^2 = \sum H_i^2 + \sum E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha$$

b. Since the H_i are mutually commuting, in a hermitian/unitary representation they are simultaneously diagonalizable. Identify basis states in a Hilbert space by their eigenvalues under the operators H_i: $|n_1, n_2, \ldots, n_r\rangle$,

$$H_i|n_1, n_2, \ldots, n_r\rangle = n_i|n_1, n_2, \ldots, n_r\rangle$$

c. For the orthogonal groups $SO(n)$, impose suitable reality conditions (i.e., $H_j \to iH_j$, etc.), choose a Hilbert space containing the state $|l, 0, \ldots, 0\rangle$ and show that the value of $\mathcal{C}^2$ on every vector (i.e., apply shift operators E_α until no new states are created) is

$$\mathcal{C}^2|\text{state}\rangle = -l(l+n-2)|\text{state}\rangle$$

The $-$ sign indicates that $SO(n)$ is compact. This spectrum reduces to the well-known spectrum $-m^2$ for $SO(2)$ (on $e^{im\phi}$) and $-l(l+1)$ for $SO(3)$ (on $Y^l_m(\theta, \phi)$).

4. **Master Analytic Representation for A_1:** The complex Lie algebra with root space diagram A_1 has two real forms

$$\begin{array}{lllll} \mathfrak{su}(2) & J_3, J_\pm & [J_3, J_\pm] = \pm J_\pm & [J_+, J_-] = +2J_3 \\ \mathfrak{su}(1,1) & K_3, K_\pm & [K_3, K_\pm] = \pm K_\pm & [K_+, K_-] = -2K_3 \end{array}$$

In problem 2 of Chapter 6 we have exploited the isomorphism between the Lie algebra $\mathfrak{su}(2)$ and bilinear combinations of creation and annihilation operators for two modes in order to construct matrix elements of the angular momentum operators. These are matrix elements of a *hermitian* representation of J_i, $i = 1, 2, 3$. Exponentials of the form $EXP(ir^k J_k)$, with r^k real, provide *unitary* representations of the compact Lie group $SU(2)$. *All* unitary irreducible representations of $SU(2)$ are finite dimensional $(2j+1)$ and are obtained in this way. In this problem we will review the construction of the UIR (unitary irreducible representations) of the compact group $SU(2)$ and will use similar methods to construct all the UIR of its analytic continuation, the noncompact Lie group $SU(1,1)$. Since the algebras are related by analytic continuation, so also are the UIR. We will begin with the analytic hermitian matrix elements for $\mathfrak{su}(2)$ and continue them to hermitian matrix elements for the analytically continued algebra $\mathfrak{su}(1,1)$.

a. Make the identifications

$$\begin{array}{ccccc} K_3 & = & \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) & = & J_3 \\ K_+ & = & i a_1^\dagger a_2 & = & i J_+ \\ K_- & = & i a_2^\dagger a_1 & = & i J_- \end{array} \tag{11.28}$$

Verify all commutation relations are satisfied.

b. Recall that in both $SU(2)$ and $SU(1,1)$, rotation about the z axis by 4π radians returns to the same group operation. Show that in any matrix representation of $SU(2)$ with J_3 diagonal, or in any matrix representation of $SU(1,1)$ with K_3 diagonal, the matrix is diagonal with matrix elements $e^{im\phi}\delta_{m'm}$. Show that the single valuedness condition under $\phi \to \phi + 4\pi$ requires that $m = \frac{1}{2}(n_1 - n_2)$ is integer or half-integer. Show that the shift operators $J_\pm$ and $K_\pm$ require that all m values in a UIR with J_3 or K_3 diagonal are either integer or half-integer.

c. Relax the assumption that all indices n_1, n_2 in the basis states $|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$ must be nonnegative integers. Construct the matrix elements of the operators $J_3, J_\pm, K_3, K_\pm$ under this relaxed assumption. Show that all commutation relations are satisfied in the representation afforded by this set of basis states.

d. Show that the matrices for J_x and J_y are also hermitian provided that

$$\begin{array}{ccc} \langle n_1 + 1, n_2 - 1 | J_+ | n_1, n_2 \rangle & = & \sqrt{(n_1+1)n_2} \\ \| & & \| \\ \langle n_1, n_2 | J_- | n_1 + 1, n_2 - 1 \rangle^* & = & \sqrt{(n_1+1)n_2}^{\,*} \end{array}$$

Show that these conditions are satisfied for

$$n_1 \geq 0 \text{ and } n_2 \geq 0 \qquad \text{or} \qquad n_1 \leq -1 \text{ and } n_2 \leq -1$$

Show that the lattice sites in quadrants I and III of Fig. 11.2, with vertices at $(0,0)$ (QI) and $(-1,-1)$ (QIII), satisfy these conditions.

e. With the identification $\left| \begin{smallmatrix} j \\ m \end{smallmatrix} \right\rangle = |n_1, n_2\rangle$, $j = \frac{1}{2}(n_1 + n_2)$, $m = \frac{1}{2}(n_1 - n_2)$, show that

$$\left\langle \begin{smallmatrix} j \\ m \pm 1 \end{smallmatrix} \right| J_\pm \left| \begin{smallmatrix} j \\ m \end{smallmatrix} \right\rangle = \sqrt{(j \mp m)(j \pm m + 1)} = \sqrt{(j + \frac{1}{2})^2 - (m \pm \frac{1}{2})^2} \tag{11.29}$$

f. Show that the operators $J_\pm$ act diagonally. In order for all states connected by successive application of these operators to remain in QI or QIII, n_1 and n_2 must be integers so that in the various quadrants the shift operators vanish on the edges as shown:

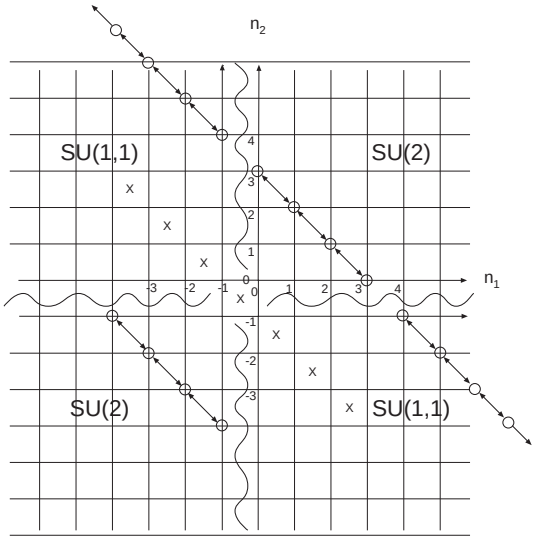

Fig. 11.2. The integer lattice in two dimensions carries representations of the algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ that exponentiate to unitary irreducible representations with careful choice of the basis set. All points in this plane are mapped to other points along diagonals of the form $n_1 + n_2 =$ constant. The subspaces of basis vectors for the unitary irreducible representations of $SU(2)$ and $SU(1,1)$ are separated by a "no mans land" defined by $-1 < n_1 < 0$ and $-1 < n_2 < 0$ (wavy lines). All points labeled x belong the the Principal series of representations of $SU(1,1)$ with $n_1 + n_2 = -\frac{1}{2} + i\beta$.

Quadrant	Operator	Edge
I	J_+	$n_2 = 0$
I	J_-	$n_1 = 0$
III	J_+	$n_1 = -1$
III	J_-	$n_2 = -1$

g. Show that the matrix elements for the shift operators $K_\pm$ in $\mathfrak{su}(1,1)$ are

$$\langle \begin{array}{c} j \\ m \pm 1 \end{array} |K_\pm| \begin{array}{c} j \\ m \end{array} \rangle = i\sqrt{(j \mp m)(j \pm m + 1)} = \sqrt{(m \pm \frac{1}{2})^2 - (j + \frac{1}{2})^2} \tag{11.30}$$

h. For $\mathfrak{su}(1,1)$ show the hermiticity condition is satisfied for all real numbers except those in QI and QIII.

i. In order to ensure that a set of states $|n_1 + k, n_2 - k\rangle$ (k integer) mapped into each other by the shift operators $K_\pm$ do not enter QI from QIV, show that the edge (lowest m) state must be $|n_1, n_2 = -1\rangle$ for $n_1 = 0, 1, 2, \ldots$. The basis states for this bounded discrete series of representations are $|\begin{matrix} j \\ m \end{matrix}\rangle = |n_1, n_2\rangle$ with $j + \frac{1}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ or $2j+1 = 0, 1, 2, 3, \ldots$ and $m = j+1, j+2, \ldots$. This is the Discrete series of representations that is bounded below: $\mathcal{D}^j_+$.

j. In order to ensure that a set of states $|n_1 + k, n_2 - k\rangle$ (k integer) mapped into each other by the shift operators $K_\pm$ do not enter QI from QII, show that the edge (highest m) state must be $|n_1 = -1, n_2\rangle$ for $n_2 = 0, 1, 2, \ldots$. The basis states for this bounded discrete series of representations are $|\begin{matrix} j \\ m \end{matrix}\rangle = |n_1, n_2\rangle$ with $j + \frac{1}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ or $2j+1 = 0, 1, 2, 3, \ldots$ and $m = -j-1, -j-2, \ldots$. This is the Discrete series of representations that is bounded above: $\mathcal{D}^j_-$.

k. Advance similar arguments to guarantee that states do not enter QIII from QIV ($\mathcal{D}^j_+$) or from QII ($\mathcal{D}^j_-$).

l. Now relax the condition that n_1 and n_2 are integers. The set of states $|n_1 + k, n_2 - k\rangle$ ($k = \ldots, -2, -1, 0, +1, +2, \ldots$) connected by $K_\pm$ carries a hermitian representation of $\mathfrak{su}(1,1)$ if one of the states falls in the shaded square with corners on the vertices of the four quadrants. If this state is $|p, q\rangle$, with $-1 \leq p, q \leq 0$ then the single-valuedness condition requires $\frac{1}{2}(p-q) =$ integer or half- integer. In the latter case, it is not possible for the matrix element in Eq. (11.30) to be real for all values of the $U(1)$ index m. Therefore $p - q = 0$ and $-1 \leq p = q \leq 0$. The states $|\begin{matrix} j \\ m \end{matrix}\rangle$ with m integer, j real and $-\frac{1}{2} \leq j + \frac{1}{2} \leq +\frac{1}{2}$ carry representations $\mathcal{D}^p$ of the Complementary series of UIR for $SU(1,1)$.

m. By setting $j + \frac{1}{2} = i\beta$ (β real) the matrix elements in Eq. (11.30) become

$$\langle \begin{matrix} j \\ m \pm 1 \end{matrix} |K_\pm| \begin{matrix} j \\ m \end{matrix} \rangle = i\sqrt{(j \mp m)(j \pm m + 1)} = \sqrt{(m \pm \frac{1}{2})^2 + \beta^2} \tag{11.31}$$

These matrix elements are always positive, both for representations with m integer and those with m half-integer. These states carry UIR belonging to the Principal series of representations of $SU(1,1)$.

n. The four series of UIR for $SU(1,1)$ are

Principal	$j+\frac{1}{2}=i\beta$	β real	$m=0,\pm1,\pm2,...$
			$m=\pm\frac{1}{2},\pm\frac{3}{2},...$
Complementary	$j+\frac{1}{2}=p$	$-\frac{1}{2}\le p\le+\frac{1}{2}$	$m=0,\pm1,\pm2,...$
Discrete, +	$2j+1=0,\pm1,\pm2...$		$m=+\lvert j\rvert+1,+\lvert j\rvert+2...$
Discrete, −	$2j+1=0,\pm1,\pm2...$		$m=-\lvert j\rvert-1,-\lvert j\rvert-2...$

Show that states with $j'<-\frac{1}{2}$ obtained by reflection through the diagonal containing the central point in the shaded square with coordinates $(n_1,n_2)=(-\frac{1}{2},-\frac{1}{2})$ support representations equivalent to those with index $j>-\frac{1}{2}$. The relation among indices is $j+\frac{1}{2}=-(j'+\frac{1}{2})$ and the relation among states is $\left|\begin{smallmatrix}j'\\m'\end{smallmatrix}\right\rangle\simeq\left|\begin{smallmatrix}j\\m=m'\end{smallmatrix}\right\rangle$.

o. Show that the following equivalences occur among representations of these four series:

Principal series	$j'=-\frac{1}{2}-i\lvert\beta\rvert\leftrightarrow j=j'^{*}=-\frac{1}{2}+i\lvert\beta\rvert$
Complementary series	$-\frac{1}{2}\le j'+\frac{1}{2}\leftrightarrow(j+\frac{1}{2})=-(j'+\frac{1}{2})<0$
Discrete series, +	$j'+\frac{1}{2}<0\leftrightarrow j+\frac{1}{2}=-(j'+\frac{1}{2})$
Discrete series, −	$j'+\frac{1}{2}<0\leftrightarrow j+\frac{1}{2}=-(j'+\frac{1}{2})$

5. A real simple Lie algebra of rank l and dimension n has basis vectors H_i and E_α. An element X in the Lie algebra is a real linear combination of these generators: $X=h^iH_i+e^\alpha E_\alpha$, with h^i,e^α real. Show that the real subalgebra spanned by the $\frac{1}{2}(n-l)$ linear combinations of the form $(E_\alpha-E_{-\alpha})/\sqrt{2}$ is the maximal compact subalgebra of this simple Lie algebra.

6. The noncompact real form $\mathfrak{sp}(p,q)$ of the symplectic algebra was constructed from the compact real form $\mathfrak{sp}(p+q)$ by "Minkowski's trick," or analytic continuation. This procedure is delicate: one must be careful of the complex unit i with quaternions. Show by more careful arguments that the result stated is correct.