9

Structure Theory for Simple Lie Algebras

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In this chapter we continue the development begun in the previous chapter. These two chapters focus on determining the structure of a Lie algebra and putting it into some canonical form. In the previous chapter we determined the types of subalgebras that every Lie algebra is constructed from. In this chapter we put the commutation relations into a standard form. This can be done for any Lie algebra. For semisimple Lie algebras this standard form has a very rigid structure whose usefulness is surpassed only by its beauty.

9.1 Objectives of This Program

In the previous chapter we studied the commutation relations of a Lie algebra through its regular representation. This study was carried out using as a tool the Cartan-Killing inner product. We carried this study about as far as possible using only that single tool. In the present chapter we introduce a second powerful tool from the theory of linear vector

spaces. This is the eigenvalue decomposition. This tool is introduced in an attempt to find standard forms for the commutation relations. If a standard form is available then the properties of a Lie algebra, as well as its identification (classification), can be made at sight.

The eigenoperator decomposition is effected by computing and studying a secular equation determined from the matrix of the regular (or any other matrix) representation of the Lie algebra. To get the most information from this study we seek the maximum number of independent roots of this equation. The decomposition of the Lie algebra into eigenoperators according to the roots of the secular equation, and the properties of these roots, can also be discussed for any Lie algebra. However, for Lie algebras with a nonsingular Cartan-Killing inner product — semisimple and simple Lie algebras — the properties of the roots are very rigidly prescribed. This leads to a very elegant set of canonical commutation relations.

In introducing an eigenvalue equation it is necessary to extend the field over which the Lie algebra is defined from the real to the complex numbers. Without this extension it is not always possible to find roots of the secular equation. This field extension has the drawback that several different Lie algebras [e.g. $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$] have the same complex extension and have their different commutation relations cast into the same canonical form. We return to this question in Chapter 11, where the problem is resolved.

9.2 Eigenoperator Decomposition – Secular Equation

It would be very useful to find vectors Z, X in the Lie algebra that obeyed the 'eigenoperator' commutation relations

$$[Z, X] = \lambda X \tag{9.1}$$

It would be even more useful if we could find a set of basis vectors for the Lie algebra which all *simultaneously* obeyed commutation relations of the eigenoperator type.

To determine operators X for which such commutation relations are possible, we write $X = \sum_{i=1}^{N} a^i X_i$, where X_i form a basis set. Then

$$\begin{aligned} \left[Z, \sum a^i X_i\right] &= \lambda \sum a^j X_j \\ \sum \sum a^i \left(R(Z)_i{}^j - \lambda \delta_i{}^j\right) X_j &= 0 \end{aligned} \tag{9.2}$$

This equation has a nonzero solution for the coefficients a^i provided the

secular equation

$$\parallel R(Z) - \lambda I_N \parallel = 0 \tag{9.3}$$

can be solved. This equation can be expanded as a polynomial in λ

$$\sum_{j=0}^{N} (-\lambda)^{N-j} \phi_j(Z) = 0 \tag{9.4}$$

where N is the dimension of the Lie algebra and its regular representation. The coefficients $\phi_j(Z)$ are homogeneous polynomials of degree j in the coefficients z^i ($Z = \sum z^i X_i$) that describe Z:

$$\phi_j(Z) \rightarrow \phi_j(z^i) \tag{9.5}$$

Example: The regular representation for the three-dimensional Lie algebra spanned by the photon creation and annihilation operators and their commutator $a^\dagger, a, I = [a, a^\dagger]$ is

$$ra^\dagger + la + \delta I \quad \begin{matrix} \text{regular} \\ \longrightarrow \\ \text{representation} \end{matrix} \quad \begin{bmatrix} 0 & l & 0 \\ 0 & 0 & 0 \\ 0 & -r & 0 \end{bmatrix} \quad \begin{matrix} a^\dagger \\ I \\ a \end{matrix} \tag{9.6}$$

With this ordering of basis vectors the regular representation does not have the structure indicated in (8.4) and Fig. 8.1 for a nilpotent algebra. The secular equation is

$$\parallel \mathfrak{Reg}(ra^\dagger + la + \delta I) - \lambda I_3 \parallel = (-\lambda)^3 = 0 \tag{9.7}$$

Strictly upper (or lower) triangular matrices have a secular equation of this form. The converse is true. If the secular equation of an $N \times N$ matrix is $(-\lambda)^N = 0$, then a basis can be found in which the matrix has strictly upper (or lower) triangular form. Therefore, there is a permutation transformation of the basis vectors that brings the regular representation of this Lie algebra to strictly upper triangular form, and the algebra is nilpotent by inspection.

Example: For $X = \sum a_i X_i \in \mathfrak{su}(2)$ the defining 2×2 matrix representation $\mathfrak{def}(X)$ and the regular 3×3 matrix representation $\mathfrak{Reg}(X)$ are

$$\mathfrak{def}(X) = \frac{1}{2} \begin{bmatrix} ia_3 & i(a_1 - ia_2) \\ i(a_1 + ia_2) & -ia_3 \end{bmatrix} \quad \mathfrak{Reg}(X) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{9.8}$$

The secular equation for the regular representation is

$$\begin{aligned} \| \mathfrak{Reg}(X) - \lambda I_3 \| &= (-\lambda)^3 + (-\lambda)(+a_1^2 + a_2^2 + a_3^2) = 0 \\ &= (-\lambda)(\lambda^2 + \phi_2(\mathbf{a})) \\ \phi_2(\mathbf{a}) &= +a_1^2 + a_2^2 + a_3^2 \end{aligned} \tag{9.9}$$

Since $\phi_2(\mathbf{a}) \geq 0$, this secular equation cannot be solved over the field of real numbers. Extension of the field from the real to the complex numbers allows factorization to find the three [= dimension of $\mathfrak{su}(2)$] roots: $\lambda = 0$, $\lambda = \pm ia$, $a^2 = +a_1^2 + a_2^2 + a_3^2$.

Example: For $Y = \sum b_i Y_i \in \mathfrak{su}(1,1)$ the defining 2×2 matrix representation $\mathfrak{def}(Y)$ and the regular 3×3 matrix representation $\mathfrak{Reg}(Y)$ are

$$\mathfrak{def}(Y) = \frac{1}{2} \begin{bmatrix} ib_3 & b_1 - ib_2 \\ b_1 + ib_2 & -ib_3 \end{bmatrix} \quad \mathfrak{Reg}(y) = \begin{bmatrix} 0 & -b_3 & -b_2 \\ b_3 & 0 & b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \tag{9.10}$$

The secular equation for the regular representation is

$$\begin{aligned} \| \mathfrak{Reg}(Y) - \lambda I_3 \| &= (-\lambda)^3 + (-\lambda)(-b_1^2 - b_2^2 + b_3^2) = 0 \\ &= (-\lambda)(\lambda^2 + \phi_2(\mathbf{b})) \\ \phi_2(\mathbf{b}) &= -b_1^2 - b_2^2 + b_3^2 \end{aligned} \tag{9.11}$$

By comparing the secular equations for $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)$, it is clear that the coefficients of the respective secular equations are 'analytic continuations' of each other. That is, under rotation of some coordinates from the real to the imaginary axis, $(a_1, a_2, a_3) \rightarrow (ib_1, ib_2, b_3)$, the coefficient $\phi_2(\mathbf{a}) = a_1^2 + a_2^2 + a_3^2$ of the secular equation for $\mathfrak{su}(2)$ maps to $\phi_2(\mathbf{b}) = -b_1^2 - b_2^2 + b_3^2$ for $\mathfrak{su}(1,1)$. This same rotation of coordinates maps the Lie algebra $\mathfrak{su}(2)$ to the Lie algebra $\mathfrak{su}(1,1)$.

The secular equation was written down for the regular representation, since it can always be constructed from the Lie algebra. A secular equation could just as easily be written down for any matrix representation of the Lie algebra. We are by and large interested in studying matrix Lie algebras, so secular equations can be written directly for the defining matrix algebras. There is a great deal of utility in this approach. First, the matrices in a matrix algebra are almost always smaller — much smaller — than the matrices of its regular representation. Second, a matrix Lie algebra contains at least as much information (certainly not less) as its regular representation.

Example: The secular equation for the defining 2×2 matrix representation of $\mathfrak{su}(2)$ in (9.8) is

$$\| \mathfrak{def}(X) - \lambda I_2 \| = \lambda^2 + (\frac{1}{2})^2(+a_1^2 + a_2^2 + a_3^2) = 0 \tag{9.12}$$

Similarly, the secular equation for the defining 2×2 matrix representation of $\mathfrak{su}(1,1)$ in (9.10) is

$$\| \mathfrak{def}(Y) - \lambda I_2 \| = \lambda^2 + (\frac{1}{2})^2(-b_1^2 - b_2^2 + b_3^2) = 0 \tag{9.13}$$

For each algebra the functional forms of the nonzero coefficient ϕ_2 in the secular equation are the same in the defining and the regular matrix representations.

9.3 Rank

The **rank**, l, of a Lie algebra is the number of independent coefficients in the secular equation of its regular representation, $\mathfrak{Reg}$. Since the number of independent roots of the secular equation is equal to the number of independent coefficients $\phi_j(z^i)$, the rank is also the number of independent roots of the secular equation. The rank is always smaller than the dimension of the Lie algebra, since there is always at least one zero root [put $X = Z$ in (9.1)]. For simple Lie algebras of dimension N, $l^2 \sim N$, so describing commutation relations in terms of rank rather than dimension effects a big simplification.

9.4 Invariant Operators

If $\phi_j(z^i)$ is a coefficient in the secular equation, the operator obtained by the symmetrized substitution

$$z^i \to X_i \qquad \phi_j(z^i) \longrightarrow \phi_j(X_i) \tag{9.14}$$

is an invariant operator: it commutes with all elements of the Lie algebra

$$[\phi_j(X_i), X_k] = 0 \tag{9.15}$$

The number of independent invariant operators ('Casimir invariants') is at least equal to the rank of the algebra, and may be as large as the dimension for a commutative algebra, where all N operators mutually commute.

Example: From the secular equation (9.9) for $\mathfrak{su}(2)$ we immediately construct a second order invariant operator that commutes with all operators in $\mathfrak{su}(2)$

$$\phi_2(\mathbf{a}) = +a_1^2 + a_2^2 + a_3^2 \longrightarrow \phi_2(X) = +X_1^2 + X_2^2 + X_3^2 \tag{9.16}$$

A similar calculation for $\mathfrak{su}(1,1)$ gives

$$\phi_2(\mathbf{b}) = -b_1^2 - b_2^2 + b_3^2 \longrightarrow \phi_2(Y) = -Y_1^2 - Y_2^2 + Y_3^2 \tag{9.17}$$

Notice that the Casimir invariant operator for $\mathfrak{su}(1,1)$ is the analytic continuation of the Casimir invariant operator for $\mathfrak{su}(2)$.

If $\mathfrak{m}$ is some matrix Lie algebra of $n \times n$ matrices, then any operator in $\mathfrak{m}$ can be written as a linear combination of matrices M_{ij}, with entry $+1$ at the intersection of the ith row and jth column and zeroes elsewhere

$$M : \sum a^i{}_j M_i{}^j \tag{9.18}$$

The coefficients of the secular equation for this algebra of $n \times n$ matrices are shown in Fig. 9.1.

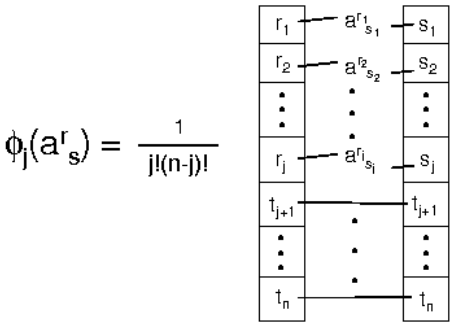

Fig. 9.1. Coefficients in the secular equation are expressed in terms of the fully antisymmetric Levi-Civita tensor on n symbols.

In this figure the vertical symbol is the Levi-Civita symbol for n dimensions (e.g., in R^3, $= \epsilon_{ijk} = +1$ for (ijk) a cyclic permutation of (123), -1 for a cyclic permutation of (321), and zero otherwise). Contracted dummy indices are connected by lines. The invariant operators for the Lie algebra of $n \times n$ matrices are shown in Fig. 9.2. Contracted dummy indices are connected by lines. The invariance of these operators depends only on the commutation relations of the Lie algebra. Therefore these invariant operators $\phi_j(X^r{}_s)$ remain invariant when the matrices

are replaced by any set of operators (cf., Chapter 6) with isomorphic commutation relations.

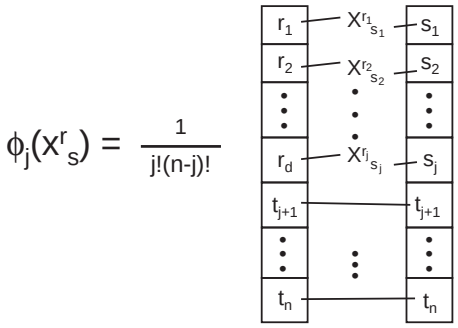

Fig. 9.2. Invariant operators $\phi_j(X)$ expressed in terms of the fully antisymmetric Levi-Civita tensor on n symbols. The invariant operators are obtained by replacing the coordinates $a^r{}_s$ by the operators $X^r{}_s$ in the coefficients ϕ_j of the secular equation. Here the general element in the Lie algebra is $X = a^r{}_s X^r{}_s$.

Example: The orthogonal groups $O(n)$ and their subgroups $SO(n)$ have Lie algebras that consist of $n \times n$ antisymmetric matrices. The secular equation is far easier to compute in the defining representation of $n \times n$ antisymmetric matrices than in the $d \times d$ (the dimension of $\mathfrak{so}(n)$ is $d = n(n-1)/2$) regular matrix representation

$$\| \mathfrak{def}(X) - \lambda I_n \| = \sum (-\lambda)^{n-j} \phi_j(X) = 0 \tag{9.19}$$

Further, the secular equation for a matrix and its transpose are equal, but since the Lie algebra consists of antisymmetric matrices, $\mathfrak{def}(X)^t = -\mathfrak{def}(X)$, and we find

$$\phi_j(X) = \phi_j(-X) = (-)^j \phi_j(X) \tag{9.20}$$

As a result, the only nonzero coefficients in the secular equation for $\mathfrak{so}(n)$ are the even coefficients. Therefore the algebra $\mathfrak{so}(n)$ has rank $[n/2]$.

Example: The second order Casimir invariant operator for $\mathfrak{so}(n)$ is obtained by setting $j = 2$ in Fig. 9.2 for the generators X_{ij} of $SO(n)$. Since $X_{ij} = -X_{ji}$, it is possible to 'rearrange' the contractions between the operators and the two different antisymmetric tensors, as shown in Fig. 9.3.

As a result, we can write for $\mathfrak{so}(n)$

$$\mathcal{C}_2[\mathfrak{so}(n)] = \sum X_{ij}^2 \tag{9.21}$$

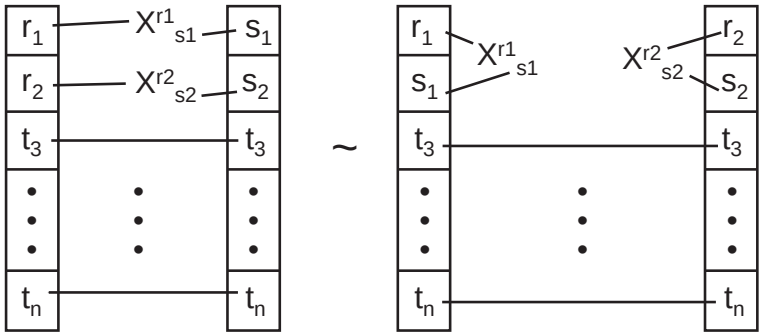

Fig. 9.3. If the operators X are antisymmetric, $X^r{}_s = -X^s{}_r$, contractions in the expressions for the Casimir operators can be rearranged as shown.

Similar 'rearrangement' arguments can be used to simplify the expressions for higher order Casimir invariant operators. For example, for $\mathfrak{so}(5)$ the fourth order Casimir operator is

$$\mathcal{C}_4[\mathfrak{so}(5)] = \sum_{i=1}^{5} v_i^2 \tag{9.22}$$

where the components of the 5-vector $\mathbf{v}$ are $v^m = \epsilon^{ijklm} X_{ij} X_{kl}$, for example

$$v^5 = \epsilon^{ijkl5} X_{ij} X_{kl} \sim X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} \tag{9.23}$$

For $\mathfrak{so}(4)$ the fourth order Casimir is a perfect square

$$\mathcal{C}_4[\mathfrak{so}(4)] = (\epsilon^{ijkl} X_{ij} X_{kl})^2 \sim (X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23})^2 \tag{9.24}$$

In general, for n even, the nth order Casimir invariant operator for $\mathfrak{so}(n)$ is a perfect square. Its square root, of order $n/2$, should be taken as an appropriate functionally independent Casimir operator. The existence of two second-order Casimir operators for $\mathfrak{so}(4)$ is another piece of evidence that this algebra is semisimple rather than simple.

9.5 Regular Elements

It is useful to choose elements Z in the Lie algebra [Eq. (9.1)] that maximize the amount of information that can be extracted from the secular equation. (At the opposite extreme, the choice $Z = 0$ is not clever since all X obey the same eigenvalue equation $[Z, X] = 0X$).

We do this by choosing a Z for which we:

(i) maximize the number of nonzero roots;
(ii) minimize the degeneracy of each nonzero root;
(iii) minimize the degeneracy of the zero root.

Such elements Z in the Lie algebra can always be found. In fact, this is a 'generic' property. 'Almost all' elements Z in the Lie algebra have this property.

As an example of this eigenoperator decomposition we treat again the six-dimensional algebra of two photon operators spanned by $\hat{n} + \frac{1}{2} = \frac{1}{2}\left\{a, a^\dagger\right\}$, $a^{\dagger 2}, a^\dagger, I = [a, a^\dagger], a, a^2$. A useful choice for Z is

$$Z = z_1(\hat{n} + \frac{1}{2}) + z_2 I \tag{9.25}$$

The secular equations for the 6×6 regular representation and the 4×4 defining matrix representations are

$$\begin{array}{lrcl} \text{regular representation} & (\lambda)^2(\lambda + 2z_1)(\lambda - 2z_1)(\lambda + z_1)(\lambda - z_1) & = & 0 \\ \text{defining representation} & (\lambda)^2(\lambda + z_1)(\lambda - z_1) & = & 0 \end{array} \tag{9.26}$$

Each secular equation has only one independent coefficient ϕ. The nontrivial coefficients of the secular equation for the regular representation are

$$\begin{array}{rcl} \phi_2(z_1, z_2) & = & -5z_1^2 \\ \phi_4(z_1, z_2) & = & 4z_1^4 \quad = \quad 4\left(-\phi_2(z_1, z_2)/5\right)^2 \end{array} \tag{9.27}$$

For the 4×4 matrix representation the one nontrivial coefficient is

$$\phi_2(z_1, z_2) = -z_1^2 \tag{9.28}$$

This is a rank-one Lie algebra since there is only one functionally independent coefficient in the secular equation. The roots of the secular equation of the regular representation are $\pm 2z_1, \pm z_1, 0, 0$ and the commutation relations can be summarized in the 'root space diagram' shown in Fig. 9.4.

From this diagram we learn

$$\begin{array}{rcl} \left[\hat{n} + \frac{1}{2}, X_{(k,0)}\right] & = & kX_{(k,0)} \\ \left[I, X_{(k,0)}\right] & = & 0X_{(k,0)} \end{array} \tag{9.29}$$

where $X_{(2,0)} = a^{\dagger 2}$, $X_{(1,0)} = a^\dagger$, $X_{(0,0)} = \hat{n} + \frac{1}{2}I, I$, $X_{(-1,0)} = a$, $X_{(-2,0)} = a^2$. We also see that if $k, l \in \{-2, -1, 0, +1, +2\}$

$$[X_{(k,0)}, X_{(l,0)}] \sim X_{(k+l,0)} \tag{9.30}$$

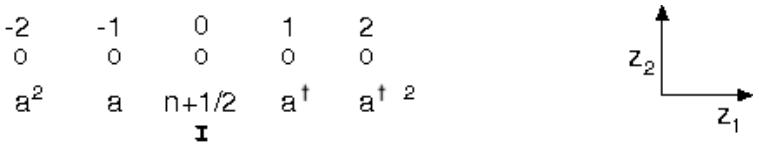

Fig. 9.4. The six operators in the two-photon algebra can be organized according to their roots, which are eigenvalues of a secular equation. Two operators have zero root.

if $k+l$ is in the set $\{-2,-1,0,+1,+2\}$ and zero otherwise. If $k+l=0$ the commutator is some linear combination of the two operators that span the subspace $(0,0)$: $\hat{n}+\frac{1}{2}$ and I.

9.6 Semisimple Lie algebras

For simple and semisimple Lie algebras the Cartan-Killing inner product is nonsingular. When this inner product is nonsingular, the decomposition of the algebra into its subspaces, one for each root of the secular equation, has additional properties. We list these properties here, providing only an occasional proof. A more complete treatment of this, the most beautiful part of Lie algebra theory, can be found elsewhere [31,44].

9.6.1 Rank

For semisimple Lie algebras the rank l is:

a. The number of independent coefficients in the secular equation;
b. The number of independent roots $\alpha_1, \alpha_2, \cdots, \alpha_l$ of the secular equation. These l independent roots can be collected together as the components of an l-dimensional vector $(\alpha_1, \alpha_2, \cdots, \alpha_l)$ in a **root space**;
c. The dimension of the subspace V_0 (which is a subalgebra) of the root space;
d. The number of independent invariant operators (Casimir operators).

9.6.2 Properties of Roots

Further, the roots have the following properties:

a. If α and β are roots with subspaces V_α and V_β in the Lie algebra, then

$$[V_\alpha, V_\beta] \subset V_{\alpha+\beta} \tag{9.31}$$

That is, the commutator of any vector in V_α with any vector in V_β is a vector in $V_{\alpha+\beta}$. If $\alpha+\beta$ is not a root, the commutator vanishes.

b. The l basis vectors $H_1, H_2, \cdots, H_l$ in the l-dimensional subspace V_0 commute:

$$[H_i, H_j] = 0 \qquad 1 \le i, j \le l \tag{9.32}$$

c. Each subspace V_α ($\alpha \neq 0$) is one-dimensional. Therefore each subspace V_α is spanned by an operator E_α that can be labeled by the root α. As a result (i.e., $[V_0, V_\alpha] \subset V_\alpha$), each H_i maps E_α into a multiple of itself

$$\begin{aligned} [H_i, E_\alpha] &= \alpha_i E_\alpha \\ [\mathbf{H}, E_\alpha] &= \alpha\ E_\alpha \end{aligned} \tag{9.33}$$

d. If α is a root, $-\alpha$ is a root. If α is a root and $c\alpha$ is a root, then $|c| = 1$. Thus, nonzero roots occur in pairs of opposite sign. In addition, the only root collinear with 0 and α is $-\alpha$.

e. The commutator of E_α and $E_{-\alpha}$ is in V_0, so can be expanded as a linear superposition of the H_i:

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i \tag{9.34}$$

f. An inner product relating α^i and α_j by $\alpha^i = h^{ij}\alpha_j$ can be introduced in this root space

$$(\alpha, \beta) = \alpha_i \beta^i = \alpha^j \beta_j = \alpha_i h^{ij} \beta_j \tag{9.35}$$

This inner product is positive definite. If the lengths of the roots are normalized so that

$$\sum_{\alpha \neq 0} \alpha_i \alpha_j = \delta_{ij} \qquad \text{or} \qquad \sum_{\alpha \neq \mathbf{0}} \alpha \cdot \alpha = \text{rank} = l \tag{9.36}$$

then $h^{ij} = \delta^{ij}$ and we can identify α^i with α_i: $\alpha^i = \alpha_i$.

g. It remains to compute

$$\begin{array}{rcll} & \to & 0 & \alpha+\beta \text{ not a root} \\ [E_\alpha, E_\beta] & \to & N_{\alpha,\beta} E_{\alpha+\beta} & \alpha+\beta \text{ a root} \\ & \to & \alpha \cdot \mathbf{H} & \alpha+\beta = 0 \end{array}$$

Three cases arise, as indicated. The only detail remaining is to determine the coefficient $N_{\alpha,\beta}$ when $\alpha+\beta$ is a nonzero root.

9.6.3 Structure Constants

To compute these coefficients we first apply the Jacobi identity to the generators $E_\alpha, E_\beta, E_\gamma$ of three nonzero roots that sum to zero

$$[[E_\alpha, E_\beta], E_\gamma] + [[E_\beta, E_\gamma], E_\alpha] + [[E_\gamma, E_\alpha], E_\beta] = 0 \tag{9.37}$$

From this we derive the symmetry

$$\begin{array}{lccccccc} \text{when} & \alpha & + & \beta & + & \gamma & = & 0 \\ \text{then} & \alpha N_{\beta,\gamma} & + & \beta N_{\gamma,\alpha} & + & \gamma N_{\alpha,\beta} & = & 0 \\ \text{and} & N_{\beta,\gamma} & = & N_{\gamma,\alpha} & = & N_{\alpha,\beta} & & \end{array} \tag{9.38}$$

Next we compute a recursion relation involving these coefficients. This is done by embedding β in a chain of roots involving α additively, as shown in Fig. 9.5. In this chain

$$\beta - m\alpha \quad \beta-(m-1)\alpha \quad \cdots \quad \beta \quad \beta+\alpha \quad \cdots \quad \beta+n\alpha$$

are all roots but

$$\begin{array}{lcl} \beta & - & (m+1)\alpha \\ \beta & + & (n+1)\alpha \end{array} \tag{9.39}$$

are not roots. By applying the Jacobi identity to roots $\alpha, \beta+k\alpha$, and $-\alpha$ we obtain the recursion relation

$$N^2_{\alpha,\beta+(k-1)\alpha} = N^2_{\alpha,\beta+k\alpha} + \alpha\cdot(\beta+k\alpha) \tag{9.40}$$

This recursion relation satisfies the boundary conditions

$$\begin{array}{rcl} N^2_{-\alpha,\beta-m\alpha} & = & 0 \\ N^2_{+\alpha,\beta+n\alpha} & = & 0 \end{array} \tag{9.41}$$

The initial condition $N_{\alpha,\beta+n\alpha}=0$ leads to

$$N^2_{\alpha,\beta+(k-1)\alpha} = (n-k+1)(\alpha\cdot\beta + \frac{1}{2}(n+k)\alpha\cdot\alpha) \tag{9.42}$$

The other boundary condition $N^2_{-\alpha,\beta-m\alpha} = N^2_{\alpha,\beta-(m+1)\alpha} = 0$ leads to

$$N^2_{\alpha,\beta-(m+1)\alpha} = (n+m+1)(\alpha\cdot\beta + \frac{1}{2}(n-m)\alpha\cdot\alpha) = 0 \tag{9.43}$$

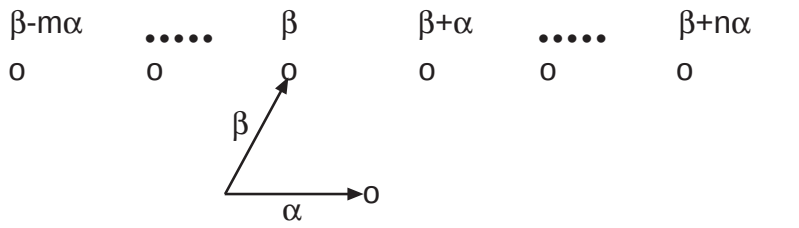

Fig. 9.5. α chain containing β. This chain is used to compute coefficients $N_{\alpha,\beta}$ in commutators $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$.

9.6.4 Root Reflections

From this we extract the following information

a. $N^2_{\alpha,\beta+k\alpha} = (n-k)(m+k+1)(\alpha\cdot\alpha)/2 \geq 0$. We use this expression because it shows clearly how the boundary conditions are imposed. We note that $\alpha\cdot\beta > 0$ when $m-n>0$ and $\alpha\cdot\beta<0$ when $m-n<0$.

b. The inner products obey

$$-n \leq \frac{2\alpha\cdot\beta}{\alpha\cdot\alpha} = -n+m \leq m \tag{9.44}$$

where m and n are nonnegative integers.

c. If β is a root, then

$$\beta' = \beta - 2\frac{\beta\cdot\alpha}{\alpha\cdot\alpha}\alpha \tag{9.45}$$

is also a root. This root is obtained by reflecting β in the hyperplane orthogonal to α.

All of the rank-2 root space diagrams are shown in Fig. 9.6. There the symmetries of root spaces under reflection and rotation may be seen.

9.7 Canonical Commutation Relations

The root space diagram encapsulates in a very convenient way all the structure constants of a semisimple Lie algebra. The basis vectors are the l (= rank) operators $\mathbf{H} = (H_1, H_2, \cdots, H_l)$ and the 'shift' operators E_α, one corresponding to each nonzero root. The root vector

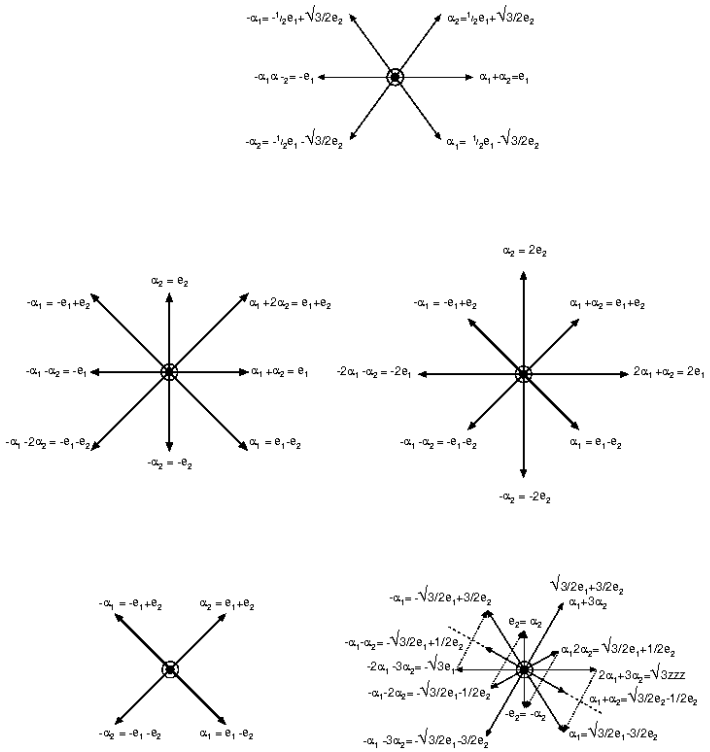

Fig. 9.6. Two-dimensional root space diagrams. Top: A_2, B_2, C_2. Bottom: D_2, G_2.

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ has l components. The commutation relations are

$$\begin{array}{rcll} [H_i, H_j] & = & 0 & \\ [\mathbf{H}, E_\alpha] & = & \alpha E_\alpha & \\ [E_\alpha, E_\beta] & = & \alpha \cdot \mathbf{H} & \alpha + \beta = 0 \\ & = & N_{\alpha,\beta} E_{\alpha+\beta} & \alpha + \beta \neq 0, \text{ a root} \\ & = & 0 & \alpha + \beta \text{ not a root} \end{array} \tag{9.46}$$

These commutation relations are subject to:

$$\begin{array}{ll} \text{normalization} & \sum_{\alpha \neq 0} \alpha \cdot \alpha = \text{rank} = l \\ & \\ \text{Jacobi} & N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha} \qquad \alpha + \beta + \gamma = 0 \\ \text{symmetry} & N_{\alpha,\beta} = -N_{-\alpha,-\beta} = -N_{\beta,\alpha} \end{array} \tag{9.47}$$

As an example of the rigidity of these commutation relations, we

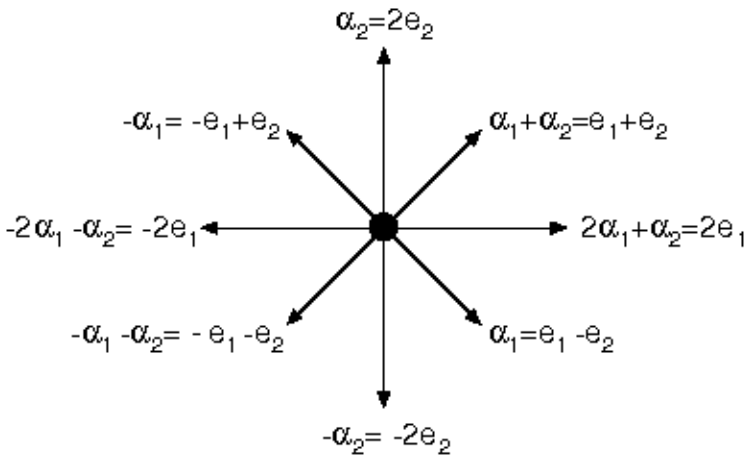

Fig. 9.7. Root space C_2. The roots are expressed in terms of a Cartesian (orthogonal) set $\mathbf{e}_1, \mathbf{e}_2$ and a fundamental set α_1, α_2.

write down the commutation relations described by the rank-2 root space C_2 shown in Fig. 9.7. If we choose orthogonal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in a two-dimensional Euclidean space, the nonzero roots for C_2 are $\pm 2\mathbf{e}_1, \pm 2\mathbf{e}_2, \pm\mathbf{e}_1 \pm \mathbf{e}_2$. The 10 basis vectors in the Lie algebra are H_i, $i = 1, 2$, and E_α, with α the eight nonzero roots. We normalize these roots by $\sum \alpha \cdot \alpha = 2$ so that

$$(\mathbf{e}_i, \mathbf{e}_j) = \frac{1}{12}\delta_{ij} \tag{9.48}$$

Under this normalization condition the commutation relations are given in Table 9.1. All commutators not explicitly shown in this table vanish. For this rank-two algebra two phases may be set arbitrarily. The two commutators at which the phase choices have been made are indicated by * in Table 9.1. Both choices are $+1$. Other phase choices (-1) give isomorphic commutation relations.

9.8 Conclusion

The structure constants for a Lie algebra have been reduced to a canonical form by studying the properties of its regular representation. Using the Cartan-Killing inner product it is possible to determine the semisimple part of a Lie algebra and its complement, the maximal solvable in-

Table 9.1. *Nonzero commutators for Lie algebras with root space C_2.*

$$
\begin{aligned}
[H_1, H_2] &= 0 \\
[\mathbf{H}, E_{\pm 2\mathbf{e}_1}] &= (\pm 2/\sqrt{12}, 0) E_{\pm 2\mathbf{e}_1} \\
[\mathbf{H}, E_{\pm 2\mathbf{e}_2}] &= (0, \pm 2/\sqrt{12}) E_{\pm 2\mathbf{e}_2} \\
[\mathbf{H}, E_{\pm \mathbf{e}_1 \pm \mathbf{e}_2}] &= (\pm/\sqrt{12}, \pm/\sqrt{12}) E_{\pm \mathbf{e}_1 \pm \mathbf{e}_2} \\
[E_{+2\mathbf{e}_1}, E_{-2\mathbf{e}_1}] &= (2/\sqrt{12}) H_1 \\
[E_{+2\mathbf{e}_2}, E_{-2\mathbf{e}_2}] &= (2/\sqrt{12}) H_2 \\
\left[E_{\pm \mathbf{e}_1 \pm \mathbf{e}_2}, E_{-(\pm \mathbf{e}_1 \pm \mathbf{e}_2)}\right] &= (1/\sqrt{12})(\pm H_1 \pm H_2) \\
\left[E_{+2\mathbf{e}_1}, E_{-(\mathbf{e}_1 + \mathbf{e}_2)}\right] &= *(1/\sqrt{6}) E_{\mathbf{e}_1 - \mathbf{e}_2} \\
\left[E_{-\mathbf{e}_1 + \mathbf{e}_2}, E_{-\mathbf{e}_1 - \mathbf{e}_2)}\right] &= (1/\sqrt{6}) E_{-2\mathbf{e}_1} \\
[E_{-\mathbf{e}_1 - \mathbf{e}_2}, E_{+2\mathbf{e}_1}] &= (1/\sqrt{6}) E_{+\mathbf{e}_1 - \mathbf{e}_2} \\
[E_{-2\mathbf{e}_1}, E_{\mathbf{e}_1 - \mathbf{e}_2}] &= (-1/\sqrt{6}) E_{-\mathbf{e}_1 - \mathbf{e}_2} \\
[E_{+\mathbf{e}_1 - \mathbf{e}_2}, E_{+\mathbf{e}_1 + \mathbf{e}_2}] &= (-1/\sqrt{6}) E_{+2\mathbf{e}_1} \\
[E_{+\mathbf{e}_1 + \mathbf{e}_2}, E_{-2\mathbf{e}_2}] &= (-1/\sqrt{6}) E_{+\mathbf{e}_1 - \mathbf{e}_2} \\
[E_{+2\mathbf{e}_2}, E_{-\mathbf{e}_1 - \mathbf{e}_2}] &= *(1/\sqrt{6}) E_{-\mathbf{e}_1 + \mathbf{e}_2} \\
[E_{-\mathbf{e}_1 - \mathbf{e}_2}, E_{+\mathbf{e}_1 - \mathbf{e}_2}] &= (1/\sqrt{6}) E_{-2\mathbf{e}_2} \\
[E_{+\mathbf{e}_1 - \mathbf{e}_2}, E_{+2\mathbf{e}_2}] &= (1/\sqrt{6}) E_{+\mathbf{e}_1 + \mathbf{e}_2} \\
[E_{-2\mathbf{e}_2}, E_{+\mathbf{e}_1 + \mathbf{e}_2}] &= (-1/\sqrt{6}) E_{+\mathbf{e}_1 - \mathbf{e}_2} \\
[E_{+\mathbf{e}_1 + \mathbf{e}_2}, E_{-\mathbf{e}_1 + \mathbf{e}_2}] &= (-1/\sqrt{6}) E_{+2\mathbf{e}_2} \\
[E_{-\mathbf{e}_1 + \mathbf{e}_2}, E_{-2\mathbf{e}_2}] &= (-1/\sqrt{6}) E_{-\mathbf{e}_1 - \mathbf{e}_2}
\end{aligned}
$$

variant subalgebra. An eigenvalue decomposition can be used to put the commutation relations of the semisimple part into a standard form.

When the algebra is simple or semisimple the commutation relations are elegantly summarized by a root space diagram. This is a simple

geometric structure in a Euclidean space of dimension l, where l is the rank of the Lie algebra. The rank is:

(i) The number of functionally independent coefficients in the secular equation;
(ii) The number of independent roots of the secular equation;
(iii) The number of Casimir invariant operators;
(iv) The dimension of the root space diagram;
(v) The number of mutually commuting operators in the Lie algebra.

We have illustrated how to extract commutation relations from a root space diagram for C_2.

In classifying simple Lie algebras by their root space diagram, we were forced to extend the field of the Lie algebra from the real to the complex numbers in order to guarantee that the secular equation had as many roots as basis vectors in the Lie algebra. In doing so, we have introduced a situation in which different algebras have the same complex extension [e.g., $\mathfrak{sl}(2;R)$ and $\mathfrak{so}(3)$ have common complex extension $\mathfrak{sl}(2;C)$]. Root spaces classify commutation relations of these complex Lie algebras. Root spaces also summarize the commutation relations for the various real subalgebras of these complex algebras — some roots α_i and structure constants will be imaginary. However, determining the real subalgebras of a complex Lie algebra is a not entirely trivial task to which we return in Chapter 10.

9.9 Problems

1. Construct the regular representation for the two photon operator algebra: $\frac{1}{2}\left\{a^\dagger, a\right\}$, $a^{\dagger^2}, a^\dagger, I, a, a^2$. Determine the secular equation for this matrix. Determine the rank of this Lie algebra.

2. Construct the 4×4 defining matrix representation and the 6×6 regular matrix representation of the Lie algebra $\mathfrak{so}(4)$. Construct the secular equation. This equation factors into two independent equations, each with one independent coefficient ϕ. Both are quadratic. Construct these coefficients. Use these to construct the two quadratic invariant operators on this semisimple Lie algebra. Show that in the canonical basis $X_{ij} = x^i\partial_j - x^j\partial_i$ $(1 \leq i < j \leq 4)$ these operators are $\mathcal{C}_2 = \sum_{i<j} X_{ij}^2$ and $\mathcal{C}_2^{'} = X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23}$.

3. The Lie algebra $\mathfrak{su}(4)$ has a 4×4 defining matrix representation and a 15×15 regular matrix representation. Show that the secular equation

of the regular representation has just three independent coefficients. Do this by showing that there is a relation between the secular equation for the regular representation and the secular equation for the defining matrix representation. What is this relation? The three independent coefficients in the secular equation for the defining representation are of degree 2, 3, 4. Construct the invariant operators on $\mathfrak{su}(4)$ of degree 2, 3, and 4.

4. For $\mathfrak{so}(2n+1)$ the invariant operators (Casimir operators) are of degree $2, 4, \ldots, 2n$. This is true also for $\mathfrak{so}(2n)$, with one difference: the invariant operator of degree $2n$ is a perfect square. Show that its square root, an invariant operator of degree n, is $\mathcal{C}_n' = \epsilon^{i_1 i_2 \ldots i_{2n}} X_{i_1 i_2} X_{i_3 i_4} \ldots X_{i_{2n-1}, i_{2n}}$. Explicitly write out $\mathcal{C}_2'$ for $\mathfrak{so}(4)$ and $\mathcal{C}_3'$ for $\mathfrak{so}(6)$. Compare your results with Fig. 9.3.

5. In Chapter 11 we will show that $\mathfrak{su}(4) = \mathfrak{so}(6)$. Both Lie algebras have invariant operators of degree 2, 3, 4. Constuct the isomorphism between these Lie algebras and their invariant operators.

6. Summarize the commutation relations satisfied by the algebra of photon operators for two modes. This algebra is 10 dimensional. It contains the four operators $a_i^\dagger a_j + \frac{1}{2}\delta_{ij}$ $(1 \leq i, j \leq 2)$ and the two pairs of three operators $a_i^\dagger a_j^\dagger$ and $a_i a_j$ $(a_i a_j = a_j a_i)$. Show that this root space diagram is isomorphic to C_2, shown in Fig. 9.7. The identification is: $a_i^\dagger a_i + \frac{1}{2} \leftrightarrow H_i$, $a_i^\dagger a_j^\dagger \leftrightarrow E_{+\mathbf{e}_i + \mathbf{e}_j}$ $(i \neq j)$, $a_i^\dagger a_j \leftrightarrow E_{+\mathbf{e}_i - \mathbf{e}_j}$ $(i \neq j)$, $a_i a_j \leftrightarrow E_{-\mathbf{e}_i - \mathbf{e}_j}$ $(i \neq j)$, $a_i^\dagger a_i^\dagger \leftrightarrow E_{+2\mathbf{e}_i}$, $a_i a_i \leftrightarrow E_{-2\mathbf{e}_i}$.

7. Repeat problem 6 for the algebra of two fermion operators for two modes. This algebra is 6 dimensional. Show that the resulting root space diagram is D_2 (Fig. 9.6). Why the difference? (Hint: $f_i^\dagger f_i^\dagger = 0$.)

8. The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic. In fact, the latter is the regular representation for the former. Choose $X, Y \in \mathfrak{su}(2)$ and compute $(X, Y) = \text{tr}\,[\mathfrak{Def}(X)\mathfrak{Def}(Y)]$ by taking the trace of the 2×2 matrices in $\mathfrak{su}(2)$ that represent X and Y. Now compute the inner product using instead the Lie algebra $\mathfrak{so}(3)$, that is, the regular matrix representation of $\mathfrak{su}(2)$: $(X, Y) = \text{tr}\,[\mathfrak{Reg}(X)\mathfrak{Reg}(Y)]$. Show that the two results are proportional. What is the proportionality constant?

9. Choose two vectors X and Y in the Lie algebra $\mathfrak{su}(n)$. Compute their inner product in the $n \times n$ defining matrix representation and in the $(n^2-1) \times (n^2-1)$ regular matrix representation. The two inner

products are proportional. What is the proportionality constant? Hint: set $Y = X$ and choose a special X, for example $X = H_1$.

10. Express the Lie algebras spanned by the following ten sets of operators in canonical form (b: boson operators; f: fermion operators; $1 \le i, j \le N$):

$b_i^\dagger b_j$	$b_i^\dagger b_j + \frac{1}{2}\delta_{ij}, b_i^\dagger b_j^\dagger, b_i b_j$	$b^\dagger, b^\dagger b, b^\dagger bb$	$b, b^\dagger b, b^\dagger b^\dagger b$
$f_i^\dagger f_j$	$f_i^\dagger f_j + \frac{1}{2}\delta_{ij}, f_i^\dagger f_j^\dagger, f_i f_j$	$x, x\partial, x\partial^2$	$\partial, x\partial, x^2\partial$
$x^i\partial_j$	$x^i\partial_j + \frac{1}{2}\delta_{ij}, x^i x^j, \partial_i\partial_j$		

11. Compute $\mathbf{R} = \frac{1}{2}\sum_{\alpha>0}\alpha$, half the sum over all positive roots, in each of the simple Lie algebras. This vector plays a major role in computing the spectrum of the quadratic Casimir operator for each of the irreducible representations of each of the simple Lie algebras. For example, for B_n, $R_i = \frac{1}{2}(2n+1) - i$ and the spectrum is

$$\mathcal{C}^2(\mathbf{M}) = (\mathbf{M}+\mathbf{R})\cdot(\mathbf{M}+\mathbf{R}) - (\mathbf{R})\cdot(\mathbf{R}) = \mathbf{M}\cdot\mathbf{M} + \mathbf{M}\cdot\mathbf{2R}$$

where $\mathbf{M}$ is the highest weight in the representation. For the $(2j+1)$ dimensional representation of $\mathfrak{so}(3)$, $\mathbf{M} = j$, $\mathbf{R} = R_1 = \frac{1}{2}$ and $\mathcal{C}^2(j) = (j+\frac{1}{2})^2 - (0+\frac{1}{2})^2 = j(j+1)$.

12. The Weyl group of reflections for a simple Lie algebra is generated by reflections in planes orthogonal to all the nonzero roots.

a. Show that the Weyl group for A_{n-1} is of order $n!$, the Weyl group for D_n is of order $2^{n-1}n!$, and the Weyl groups for B_n and C_n are of order $2^n n!$.

b. Show that the product of the degrees of the functionally independent coefficients in the secular equation for each of these algebras is equal to the order of the Weyl group.

c. Show that the product of the degrees of the Casimir operators for each of these algebras is equal to the order of the Weyl group.

13. Compute the dimensions of each of the classical Lie algebras as a function of the rank, and show

$$\frac{\dim(\mathfrak{g})}{\{\text{rank}(\mathfrak{g})\}^2} =$$

ratio	$n \to \infty$	Algebra
$1 + \frac{2}{n}$	1	A_n
$2 - \frac{1}{n}$	2	D_n
$2 + \frac{1}{n}$	2	B_n, C_n

14. Multilinear operations can be defined on a matrix Lie algebra by

$$(A_1, A_2, \cdots, A_r)_{\mathfrak{Reg}} = \text{tr } \mathfrak{Reg}(A_1)\mathfrak{Reg}(A_2)\cdots\mathfrak{Reg}(A_r)$$

A multilinear operator can be defined similarly in other representations as well: for example, the defining representation.

a. Show

$$\frac{(A_1, A_2, \cdots, A_r)_{\mathfrak{Reg}}}{f_r(\mathfrak{Reg})} = (A_1, A_2, \cdots, A_r) = \frac{(A_1, A_2, \cdots, A_r)_\Gamma}{f_r(\Gamma)}$$

where Γ is some irreducible representation of the Lie algebra. This relation defines the *index* $f_r(\Gamma)$.

b. Show

$$\frac{f_r(\Gamma)}{f_r(\text{def})} = \frac{\text{tr } (\Gamma(A))^r}{\text{tr } (\text{def}(A))^r} = \frac{\dim(\Gamma)\ \mathcal{C}^r(\Gamma)}{\dim(\text{def})\ \mathcal{C}^r(\text{def})}$$

In this expression $\mathcal{C}^r$ is the value of the rth Casimir invariant in the representation indicated.

c. For $\mathfrak{su}(2)$

$$f_2(j) = \frac{1}{6}\{(2j)(2j+1)(2j+2)\}\, f_2(j = \frac{1}{2})$$

15. The matrix Lie algebras $\mathfrak{so}(2n), \mathfrak{so}(2n+1), \mathfrak{sp}(2n)$ have the form $\sum_{ij} a^{ij} M_{ij}$, where: M_{ij} is a square matrix with $+1$ in the ith row and jth column and zeroes elsewhere; M is $2n \times 2n$ for $\mathfrak{so}(2n), \mathfrak{sp}(2n)$ and $(2n+1) \times (2n+1)$ for $\mathfrak{so}(2n+1)$; and suitable reality restrinctions are imposed on the coefficients a^{ij}.

a. What are the conditions on a^{ij} for each matrix Lie algebra?

b. Write down the coefficients $\phi_r(a^{ij})$ that occur in the secular equation for each of these matrix Lie algebras.

c. Show that all odd coeffieicnts $\phi_r(a^{ij})$ vanish for each of these matrix Lie algebras.

d. Express the even coefficients in terms of the Levi-Civita skew tensors $\epsilon_{i_1 i_2 \ldots i_l}$ $(l = 2n, 2n+1, 2n)$.

e. Show that the even coefficients are all functionally independent.

f. Conclude that each of these three matrix Lie algebras has rank n.

g. Show that $\phi_{2n}(a^{ij})$ is a perfect square for $\mathfrak{so}(2n)$; write down its square root; show that it is of degree n.

16. Replace the scalar parameters θ_i in the 3×3 regular representation of $\mathfrak{so}(3)$ or $\mathfrak{su}(2)$ by the corresponding operators:

$$M = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} \longrightarrow \mathcal{M} = \begin{bmatrix} 0 & J_3 & -J_2 \\ -J_3 & 0 & J_1 \\ J_2 & -J_1 & 0 \end{bmatrix}$$

a. Show tr $M^2 = -2\theta \cdot \theta$.
b. Show tr $\mathcal{M}^2 = -2\mathbf{J} \cdot \mathbf{J}$.
c. Show $\left[\mathbf{J}, \text{tr } \mathcal{M}^2\right] = 0$.
d. Show tr $\mathcal{M}^{2n+1} = 0$ and tr $\mathcal{M}^{2n} = (-2)^n(\mathbf{J} \cdot \mathbf{J})^n$.

17. **Casimir Covariants:** A semisimple Lie algebra has basis vectors X_i that satisfy commutation relations $[X_i, X_j] = C_{ij}{}^k X_k$. There are two linear vector spaces, $V^{(1)}$ and $V^{(2)}$, that carry irreducible representations of this Lie algebra: $X_i \to \Gamma^{(1)}(X_i) = Y_i$ and $X_i \to \Gamma^{(2)}(X_i) = Z_i$. Show that the Casimir covariant $g^{ij}Y_iZ_j$ commutes with $(Y+Z)_k$ (more accurately, with $\Gamma^{(1)}(X_i) \otimes I_{\dim\ V^{(2)}} + I_{\dim\ V^{(1)}} \otimes \Gamma^{(2)}(X_i)$).

18. The Cayley-Hamilton theorem guarantees that a polynomial or analytic function of a square $n \times n$ matrix X can be expressed as a finite polynomial in the first n powers of X, starting at $X^0 = I_n$:

$$f(X) = f_0 I_n + f_1 X^1 + f_2 X^2 + \cdots + f_{n-1} X^{n-1} = \sum_{j=0}^{j=n-1} f_j X^j$$

The challenge is to compute the coefficients f_j in this expansion.

a. Show that each coefficient f_j is a function of the invariants of the matrix X.

b. Show that the invariants can variously be chosen as either the independent eigenvalues $\lambda_i(X)$ or the independent coefficients $\phi_i(X)$ of the secular equation for X.

c. Show that the Cayley-Hamilton expansion simplifies considerably if the matrix X is chosen as generic diagonal. In fact it reduces to

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} f(\lambda_1) \\ f(\lambda_2) \\ \vdots \\ f(\lambda_n) \end{bmatrix}$$

The square matrix on the left is a vanderMonde matrix.

d. Compute $e^{i\phi J_z}$ for the $(2j+1)$ dimensional matrix representations

of $SU(2)$ by computing the vanderMonde matrices. Show that for $j = \frac{1}{2}, 1, \frac{3}{2}, 2$ the resulting matrices are

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \frac{3}{2} & (\frac{3}{2})^2 & (\frac{3}{2})^3 \\ 1 & \frac{1}{2} & (\frac{1}{2})^2 & (\frac{1}{2})^3 \\ 1 & -\frac{1}{2} & (-\frac{1}{2})^2 & (-\frac{1}{2})^3 \\ 1 & -\frac{3}{2} & (-\frac{3}{2})^2 & (-\frac{3}{2})^3 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -2 & 4 & -8 & 16 \end{bmatrix} \begin{bmatrix} f_0 \\ (i\phi)^1 f_1 \\ (i\phi)^2 f_2 \\ (i\phi)^3 f_3 \\ (i\phi)^4 f_4 \end{bmatrix} = \begin{bmatrix} e^{2i\phi} \\ e^{i\phi} \\ 1 \\ e^{-i\phi} \\ e^{-2i\phi} \end{bmatrix}$$

e. Invert each of these vanderMonde matrices and determine the functions $f_j(\phi)$ in the expansions of e^X for $X \in \mathfrak{su}(2)$. In particular, show

$(i\phi)^j f_j$	Representation			
	$\frac{1}{2}$	1	$\frac{3}{2}$	2
	2	3	4	5
f_0	$\cos(\phi/2)$	1	$\frac{9}{8}\cos(\frac{\phi}{2}) - \frac{1}{8}\cos(\frac{3\phi}{2})$	1
$(i\phi)^1 f_1$	$2i\sin(\phi/2)$	$i\sin(\phi)$	$\frac{9i}{4}\sin(\frac{\phi}{2}) - \frac{i}{12}\sin(\frac{3\phi}{2})$	$\frac{i}{3}\sin(\phi) - \frac{i}{6}\sin(2\phi)$
$(i\phi)^2 f_2$		$\cos(\phi) - 1$	$-\frac{1}{2}\cos(\frac{\phi}{2}) + \frac{1}{2}\cos(\frac{3\phi}{2})$	$-\frac{5}{4} + \frac{1}{3}\cos(\phi) - \frac{1}{12}\cos(2\phi)$
$(i\phi)^3 f_3$			$-i\sin(\frac{\phi}{2}) + \frac{i}{3}\sin(\frac{3\phi}{2})$	$-\frac{i}{3}\sin(\phi) + \frac{i}{6}\sin(2\phi)$
$(i\phi)^4 f_4$				$\frac{1}{4} - \frac{1}{3}\cos(\phi) + \frac{1}{12}\cos(2\phi)$

f. Recover the two well known expansions for $j = \frac{1}{2}$ and $l = 1$:

$$j = \tfrac{1}{2}: \quad e^X = \cos(\frac{\phi}{2})I_2 + \frac{\sin(\phi/2)}{\phi/2}X$$

$$l = 1: \quad e^X = I_3 + \frac{\sin(\phi)}{\phi}X + \frac{1-\cos(\phi)}{\phi^2}X^2$$