6

Operator Algebras

Contents

Lie algebras of matrices can be mapped onto Lie algebras of operators in a number of different ways. Three useful matrix algebra to operator algebra mappings are described in this chapter.

6.1 Boson Operator Algebras

It is possible to construct useful operator algebras from Lie algebras. An operator Lie algebra can be constructed from a Lie algebra of $n \times n$ matrices by introducing a set of n independent boson creation ($b_i^\dagger$) and annihilation (b_j) operators that obey the commutation relations

$$[b_i, b_j^\dagger] = I\delta_{ij} \tag{6.1}$$

with all other commutators (e.g. $[b_i, b_j], [b_i^\dagger, b_j^\dagger], [b_i, I], [b_j^\dagger, I]$) equal to zero. The operator algebra is constructed from the matrix algebra by associating to each matrix A the operator $\mathcal{A}$ that is a linear combination of creation and annihilation operators:

$$A \to \mathcal{A} = b^\dagger A b = \sum_i \sum_j b_i^\dagger A_{ij} b_j \tag{6.2}$$

The matrices and their associated operators have isomorphic commutation relations

$$
\begin{aligned}
[\mathcal{A}, \mathcal{B}] &= \left[b_i^\dagger A_{ij} b_j, b_r^\dagger B_{rs} b_s\right] \\
&= A_{ij} B_{rs} \left[b_i^\dagger b_j, b_r^\dagger b_s\right] \\
&= A_{ij} B_{rs} \left(b_i^\dagger \delta_{jr} b_s - b_r^\dagger \delta_{si} b_j\right) \\
&= b_i^\dagger A_{ij} B_{js} b_s - b_r^\dagger B_{rs} A_{sj} b_j \\
&= b_i^\dagger \left[A, B\right]_{ij} b_j \\
&= \mathcal{C}
\end{aligned}
\tag{6.3}
$$

where $[A, B] = C$. This argument is invertible. An algebra of operators bilinear in boson creation and annihilation operators for n independent modes has an isomorphic $n \times n$ matrix algebra (or matrix representation)

$$[A, B] = C \Leftrightarrow [\mathcal{A}, \mathcal{B}] = \mathcal{C} \qquad \mathcal{A} = \sum_{ij} b_i^\dagger A_{ij} b_j \tag{6.4}$$

Remark: The $2n+1$ operators $b_i, b_j^\dagger, I$ $(1 \leq i, j \leq n)$ span the Heisenberg algebra.

6.2 Fermion Operator Algebras

The success of the calculation above does not depend on the boson commutation relations (6.1). It depends, rather, on the commutation relations of bilinear products of these operators

$$[b_i^\dagger b_j, b_r^\dagger b_s] = b_i^\dagger b_s \delta_{jr} - b_r^\dagger b_j \delta_{si} \tag{6.5}$$

Any set of operators X_{ij} that satisfies isomorphic commutation relations

$$[X_{ij}, X_{rs}] = X_{is} \delta_{jr} - X_{rj} \delta_{si} \tag{6.6}$$

can be used in place of the bilinear combinations $b_i^\dagger b_j$:

$$A \to \mathcal{A} = \sum_{ij} A_{ij} X_{ij} \tag{6.7}$$

Another useful set of operators with this property is obtained from the fermion creation $(f_i^\dagger)$ and annihilation (f_j) operators for n independent modes. These operators do not even satisfy commutation relations. Rather, they satisfy **anticommutation relations**

$$\left\{f_i, f_j^\dagger\right\} = f_i f_j^\dagger + f_j^\dagger f_i = I \delta_{ij} \tag{6.8}$$

with all other bilinear anticommutators (e.g. $\{f_i, f_j\}, \left\{f_i^\dagger, f_j^\dagger\right\}$) equal to zero. Bilinear combinations of fermion operators satisfy *commutation relations* of the form (6.6), for

$$\begin{aligned}
\left[f_i^\dagger f_j, f_r^\dagger f_s\right] &= f_i^\dagger f_j f_r^\dagger f_s - f_r^\dagger f_s f_i^\dagger f_j \\
&= f_i^\dagger(\delta_{jr} - f_r^\dagger f_j)f_s - f_r^\dagger(\delta_{is} - f_i^\dagger f_s)f_j \\
&= f_i^\dagger f_s \delta_{jr} - f_r^\dagger f_j \delta_{si}
\end{aligned} \tag{6.9}$$

As a result, matrix Lie algebras can be associated with bilinear products of either boson or fermion operators.

$$[A, B] = C \Leftrightarrow [\mathcal{A}, \mathcal{B}] = \mathcal{C} \qquad \mathcal{A} = \sum_{ij} f_i^\dagger A_{ij} f_j \tag{6.10}$$

These two matrix algebra $\rightarrow$ operator algebra mappings are useful for constructing particular classes of representations for the unitary group $U(n)$ and its subgroup $SU(n)$. The mapping to a boson operator algebra greatly simplifies the construction of the symmetric representations of $U(n)$. The mapping to a fermion operator algebra greatly simplifies the construction of the antisymmetric representations of $U(n)$. A closely related mapping allows an elegant construction of the spin representations of the orthogonal groups.

6.3 First Order Differential Operator Algebras

Yet another useful set of operators that satisfies the commutation relations (6.6) are the first order differential operators

$$X_{ij} \rightarrow x_i \partial_j = x_i \frac{\partial}{\partial x_j} \tag{6.11}$$

Then

$$[A, B] = C \Leftrightarrow [\mathcal{A}, \mathcal{B}] = \mathcal{C} \qquad \mathcal{A} = \sum_{ij} x_i A_{ij} \partial_j = \sum_{ij} A_{ij} X_{ij} \tag{6.12}$$

To illustrate the use of this operator combination, we treat the matrix algebra $\mathfrak{so}(3)$ of the orthogonal group $SO(3)$

$$\mathfrak{so}(3) = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} = \theta \cdot \mathbf{L} \tag{6.13}$$

The operator algebra is

$$\left(\begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \right) \left[\begin{array}{ccc} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{array} \right] \left[\begin{array}{c} \partial_1 \\ \partial_2 \\ \partial_3 \end{array} \right] = \theta{\cdot}\mathcal{L} \tag{6.14}$$

where $\mathcal{L}_1 = x_2\partial_3 - x_3\partial_2$, $\mathcal{L}_2 = x_3\partial_1 - x_1\partial_3$, $\mathcal{L}_3 = x_1\partial_2 - x_2\partial_1$. The two algebras have isomorphic commutation relations

$$[L_i, L_j] = -\epsilon_{ijk}L_k \qquad [\mathcal{L}_i, \mathcal{L}_j] = -\epsilon_{ijk}\mathcal{L}_k \tag{6.15}$$

where L_i are 3×3 matrices and $\mathcal{L}_i$ are first order differential operators.

As another example, we treat the Lie algebra for the group $E(2) = ISO(2)$ of rigid motions (translations and rotations) in the x-y plane, whose matrix algebra may be taken in the form

$$\left[\begin{array}{ccc} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ t_1 & t_2 & 0 \end{array} \right] = \theta L_z + t_i T_i \tag{6.16}$$

This describes rotations about an axis perpendicular to the x-y plane through an angle θ and displacements in the x and y directions by t_1 and t_2. The associated operator algebra is

$$\left(\begin{array}{ccc} x_1 & x_2 & 1 \end{array} \right) \left[\begin{array}{ccc} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ t_1 & t_2 & 0 \end{array} \right] \left[\begin{array}{c} \partial_1 \\ \partial_2 \\ 1 \end{array} \right] = \theta\mathcal{L}_z + t_i\mathcal{T}_i \tag{6.17}$$

where $\mathcal{L}_z = x_1\partial_2 - x_2\partial_1$ and $\mathcal{T}_i = \partial_i$. The matrix algebra and operator algebra have isomorphic commutation relations.

Differential operator realizations of Lie algebras come about in a natural way. This is illustrated by two simple examples. The general procedure can easily be inferred from these examples. Both involve the group of affine transformations of the real line parameterized by points (a, b) in R^2 as follows

$$(a, b) \rightarrow \left[\begin{array}{cc} e^a & b \\ 0 & 1 \end{array} \right] \tag{6.18}$$

Imagine a function defined for every point p in R^1. Once a coordinate system S is chosen a coordinate, $x(p)$, can be introduced and the function can be written explicitly as a function of x

$$\begin{array}{ccc} & f(p) & \\ \downarrow & & \downarrow \\ f_S[x(p)] & = & f_{S'}[x'(p)] \end{array} \tag{6.19}$$

If a new coordinate system S' is chosen, the value of the function at p remains unchanged but the new coordinate of p, $x'(p)$, is different. Therefore the functions f_S and $f_{S'}$ must be different. We ask: How is $f_{S'}$ related to f_S?

To answer this question, assume $x'(p)$ and $x(p)$ are related by an infinitesimal group transformation

$$\left[\begin{array}{c} x' \\ 1 \end{array}\right] = \left[\begin{array}{cc} 1+da & db \\ 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ 1 \end{array}\right] \tag{6.20}$$

Then

$$f_{S'}[x'(p)] = f_S[x(x'(p))] \tag{6.21}$$

We solve for x in terms of x' by inverting the linear relation (6.20)

$$\begin{aligned} f_{S'}[x'(p)] &= f_S[x'(1-da)-db] \\ &= f_S[x'] - da\; x' \frac{\partial f_S}{\partial x'} - db\; \frac{\partial f_S}{\partial x'} \end{aligned} \tag{6.22}$$

The infinitesimal generators that transform the function at p are

$$\mathcal{X}_a = -x' \frac{\partial}{\partial x'} \qquad \mathcal{X}_b = -\frac{\partial}{\partial x'} \tag{6.23}$$

These operators have commutation relations that are isomorphic with those of the original matrix group

$$[X_a, X_b] = X_b \Leftrightarrow [\mathcal{X}_a, \mathcal{X}_b] = \mathcal{X}_b \tag{6.24}$$

As a second example we consider functions $G(x,y)$ defined on the plane R^2 that parameterizes the affine group. By repeating the arguments above

$$G_{S'}(x', y') = G_S(x, y) \tag{6.25}$$

where (x', y') and (x, y) are related by

$$\left[\begin{array}{cc} x' & y' \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} 1+da & db \\ 0 & 1 \end{array}\right] \left[\begin{array}{cc} x & y \\ 0 & 1 \end{array}\right] \tag{6.26}$$

Inverting the infinitesimal transformation, we have

$$\begin{aligned} G_{S'}(x', y') &= G_S[x = (1-da)x', y = (1-da)y' - db] \\ &= G_S(x', y') + \left\{ da \left(-x' \frac{\partial}{\partial x'} - y' \frac{\partial}{\partial y'} \right) + db \left(-\frac{\partial}{\partial y'} \right) \right\} G_S(x', y') \end{aligned} \tag{6.27}$$

The two infinitesimal generators are

$$\begin{aligned} \mathcal{X}_a &= -x' \partial/\partial x' - y' \partial/\partial y' \\ \mathcal{X}_b &= -\partial/\partial y' \end{aligned} \tag{6.28}$$

The commutation relations are preserved

$$[X_a, X_b] = X_b \Leftrightarrow [\mathcal{X}_x, \mathcal{X}_b] = \mathcal{X}_b \tag{6.29}$$

These two examples serve to demonstrate that a single matrix algebra can have many different operator realizations.

Remark: In the example above we have adopted the "passive" interpretation of group action. That is, the coordinates of a point changed by virtue of a choice of a different coordinate system, but the value of the function did not. Therefore the particular form of the function was required to change. There is another interpretation of the group action — the "active" interpretation. In this interpretation the group operation defines a new function at the initial point in accordance with

$$f_{S'}[x(p)] = f_S[x'(p)] \tag{6.30}$$

(c.f., Eq. (6.19)). Infinitesimal generators for changes in the function under the active interpretation can be computed. They are exactly the same as those computed for the passive interpretation, except for a sign change. This sign difference is encountered in the theory of rotating bodies as the difference in commutation relations for the generators of rotation in a laboratory-fixed frame and a body-fixed frame.

The "active" and "passive" interpretations of group operations are related by the Equivalence Principle (cf., Sec. 14.2).

6.4 Conclusion

Matrix algebra to operator algebra isomorphisms are easily constructed by associating to each matrix A in a matrix Lie algebra an operator $\mathcal{A} = \sum_i \sum_j A_{ij} X_{ij}$. If the operators X_{ij} obey the simple commutation relations (6.6), the commutation relations of the matrix Lie algebra and the operator algebra are isomorphic: $[A, B] = C \Leftrightarrow [\mathcal{A}, \mathcal{B}] = \mathcal{C}$. Under these conditions, complicated commutators in an operator algebra can be replaced by simpler commutators in the matrix algebra. These results extend to the respective Lie groups: products of exponentials of operators can be replaced by products of exponentials of the corresponding matrices with a little care: $e^A e^B = e^D \Leftrightarrow e^{\mathcal{A}} e^{\mathcal{B}} = e^{\mathcal{D}}$.

6.5 Problems

1. Bilinear products involving one creation and one annihilation operator for two modes generate a four-dimensional Lie algebra with basis vectors $a_i^\dagger a_j$, $1 \leq i, j \leq 2$.

a. Show that $\hat{n} = a_1^\dagger a_1 + a_2^\dagger a_2$ commutes with all the operators in this set.

b. If $\hat{n}$ is chosen as one basis vector in this four dimensional space, the remaining three operators can be chosen as $a_1^\dagger a_1 - a_2^\dagger a_2$, $a_1^\dagger a_2$, and $a_2^\dagger a_1$. Construct their commutation relations.

c. These calculations simplify considerably under the operator to matrix mapping

$$\begin{array}{cccc} a_1^\dagger a_1 + a_2^\dagger a_2 & a_1^\dagger a_1 - a_2^\dagger a_2 & a_1^\dagger a_2 & a_2^\dagger a_1 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array}$$

d. Show that the three operators $\frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$, $a_1^\dagger a_2$, and $a_2^\dagger a_1$ satisfy commutation relations isomorphic to the comutation relations of the angular momentum algebra $J_z, J_\pm$. In particular, show

$$\begin{array}{rcllcl} J_z & = & \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) & & & \\ J_x & = & \frac{1}{2}(J_+ + J_-) & & = & \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1) \\ J_y & = & \frac{1}{2i}(J_+ - J_-) & & = & \frac{1}{2i}(a_1^\dagger a_2 - a_2^\dagger a_1) \end{array}$$

e. Evaluate $J^2 = J_x^2 + J_y^2 + J_z^2$ in terms of the creation and annihilation operators, and show

$$J^2 = (\frac{1}{2}\hat{n})(\frac{1}{2}\hat{n} + 1)$$

2. **Schwinger representation of angular momentum**. Introduce two independent modes. Assume that the quantum state of mode i ($i = 1, 2$) is $|n_i\rangle$, where n_i is the number of excitations in mode i. Assume also that the creation and annihilation operators $a_i^\dagger$ and a_i act on state $|n_i\rangle$ in the usual way:

$$a_i^\dagger |n_i\rangle = \sqrt{n_i + 1}\ |n_i + 1\rangle \qquad a_i |n_i\rangle = \sqrt{n_i}\ |n_i - 1\rangle$$

Choose as a set of basis vectors the direct product states $|n_1\rangle \otimes |n_2\rangle = |n_1, n_2\rangle$. Define

$$| \begin{array}{c} j \\ m \end{array} \rangle = |n_1, n_2\rangle \qquad j = \frac{1}{2}(n_1 + n_2), \quad m = \frac{1}{2}(n_1 - n_2)$$

a. Identify the lattice sites in Fig. 6.1 with the states $|n_1, n_2\rangle = |jm\rangle$, the diagonal operator $\frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$ with the operator J_z, and the shift operators $a_1^\dagger a_2, a_2^\dagger a_1$ with J_+ and J_-.

b. Show that the four operators $a_i^\dagger a_j$ leave invariant the sum $n_1 + n_2$.

c. $J^2|jm\rangle = j(j+1)|jm\rangle$.

d. $J_z|jm\rangle = m|jm\rangle$.

e. $J_+|jm\rangle = a_1^\dagger a_2|n_1, n_2\rangle = \sqrt{n_1+1}\ \sqrt{n_2}\ |n_1+1, n_2-1\rangle = |j, m+1\rangle\sqrt{j+m+1}\sqrt{j-m}$.

f. $J_-|jm\rangle = a_2^\dagger a_1|n_1, n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\ |n_1-1, n_2+1\rangle = |j, m-1\rangle\sqrt{j+m}\sqrt{j-m+1}$.

g. $J_\pm|jm\rangle = |j, m\pm 1\rangle\sqrt{(j\pm m+1)(j\mp m)}$. NB: $J_+|j,j\rangle = 0,\ \ J_-|j,-j\rangle = 0$.

h. $\langle j'm'|J_\pm|jm\rangle = \sqrt{(j'\pm m')(j\mp m)}\ \ \delta_{j'j}\ \delta_{m',m\pm1}$.

3. Basis vectors in the Lie algebra $\mathfrak{u}(3)$ for the group $U(3)$ have commutation relations that are isomorphic to the commutation relations of the nine boson operators $a_i^\dagger a_j$, $1 \le i,j \le 3$. Choose a set of basis vectors for a matrix representation of this algebra of the form $|n_1, n_2, n_3\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle$, where for example $b_i|n_i\rangle = |n_i - 1\rangle\sqrt{n_i}$, etc.

a. Show $N = \sum_{i=1}^3 n_i$ is not changed by the action of any of the nine operators in this set.

b. Show that the dimension, D, of this representation is $D = (N + 3-1)!/N!(3-1)!$. This is the number of ways 3 nonnegative integers can be chosen whose sum is N (Bose-Einstein counting problem). In higher dimensions (n) replace $3 \rightarrow n$. D is also the number of monomials of degree N in the Taylor series expansion of a function $f(x_1, x_2, \ldots, x_n)$ of n variables.

c. Compute the matrix elements of all operators $b_i^\dagger b_j$ in this representation:

$$\langle n_1', n_2', n_3'|b_i^\dagger b_j|n_1, n_2, n_3\rangle \tag{6.31}$$

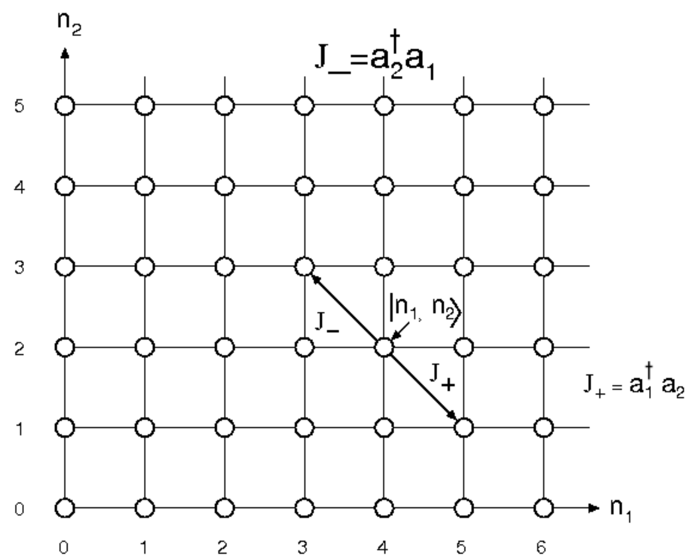

Fig. 6.1. Identification of the angular momentum operators with operators for two boson modes simplifies computation of the angular momentum matrix elements.

d. Is there some operator in the Lie algebra that maps to the identity matrix, I_D, in this representation?

$$\langle n_1', n_2', n_3'|\mathcal{O}|n_1, n_2, n_3\rangle = I_D \delta_{n_1',n_1}\delta_{n_2',n_2}\delta_{n_3',n_3} \tag{6.32}$$

What is $\mathcal{O}$?

4. Repeat the steps of Problem 3, replacing the boson operators $b_i^\dagger b_j$ by Fermion operators $f_i^\dagger f_j$. What is now the dimension of this representation?

5. Construct operators $d, d^\dagger$ defined formally from the standard creation and annihilation operators $a, a^\dagger$ as follows:

$$\begin{bmatrix} d \\ d^\dagger \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix}$$

a. Show that if the new operators $d, d^\dagger$ are to satisfy standard commutation relations $[d, d^\dagger] = 1$ and $[d, d] = [d^\dagger, d^\dagger] = 0$, the four matrix elements must satisfy $AD - BC = 1$.

b. Argue that the commutation relations are unvariant under the group $Sp(2;R) = SL(2;R)$.

c. Show that under $Sp(2;R)$, linear combinations of the coordinate and differential operators x, ∂ preserve the commutation relations. In particular, show that

$$\begin{bmatrix} a \\ a^{\dagger} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \partial \\ x \end{bmatrix}$$

preserve commutation relations.

d. Replace $a \to (a_1, a_2, \ldots, a_n)$ and similarly for $a^{\dagger}$ and their images $d, d^{\dagger}$ under some linear transformation as given above, with A, B, C, D now $n \times n$ matrices. Determine the conditions on these $n \times n$ matrices under which the structure of the commutation relations is preserved. In particular, show

$$AD^t - BC^t = I_n \qquad AB^t = BA^t \qquad CD^t = DC^t$$

Show that these transformations belong to the Lie group $Sp(2n;R)$.

6. The N-dimensional isotropic harmonic oscillator has hamiltonian

$$\mathcal{H} = \hbar\omega \sum_{i=1}^{N} (a_i^{\dagger} a_i + \frac{1}{2})$$

and eigenstates $|n_1, n_2, \ldots, n_N\rangle$.

a. Show that the degeneracy of the multiplet containing n quanta, with energy $\hbar\omega(n + \frac{N}{2})$ is $\deg(N, n) = (n + N - 1)!/n!(N-1)!$. This solution to the Bose-Einstein counting problem is exactly equal to the number of coefficients of degree n in the Taylor series expansion of a function of N variables: $f(x_1, x_2, \ldots, x_N)$.

b. Show that the symmetry group of this hamiltonian has Lie algebra spanned by the N^2 operators $a_i^{\dagger} a_j$. This is isomorphic to the Lie algebra $\mathfrak{u}(N)$. Since $\left[\mathcal{H}, a_i^{\dagger} a_j\right] = 0$, this algebra is a direct sum of a simple Lie algebra, $\mathfrak{su}(N)$, plus the one dimensional algebra spanned by $\mathcal{H}$.

c. If the generators $a_i^{\dagger} a_j$ that span the invariance algebra are supplemented with the single creation and annihilation operators $a_i^{\dagger}$ and a_j, as well as their commutator I, the resulting set of operators closes to form an $(N+1)^2$ dimensional Lie algebra that is nonsemisimple. This is called the *spectrum generating algebra* of the isotropic harmonic oscillator. Show that there is a sequence of operations drawn from this algebra that transform any state in a multiplet with n excitations to any state in a multiplet with n' excitations.

8. The set of matrices $R, S, T, U, \dots$ belong to a Lie algebra of $n \times n$ matrices, $a^\dagger = (a_1^\dagger, a_2^\dagger, \dots, a_n^\dagger)$ is a row vector of creation operators for n boson modes, and a is its adjoint, a column vector of annihilation operators. Define $\mathcal{R} = a^\dagger R a = a_i^\dagger R_{ij} a_j$,and similarly for $\mathcal{S}, \mathcal{T}, \mathcal{U}, \dots$.

a. $[R, S] = T \Leftrightarrow [\mathcal{R}, \mathcal{S}] = \mathcal{T}$
b. $e^R e^S = e^U \Leftrightarrow e^{\mathcal{R}} e^{\mathcal{S}} = e^{\mathcal{U}}$

9. The Rodriguez formula is often used to generate the Hermite polynomials.

$$H_n(x) = e^{x^2} \left(- \frac{d}{dx} \right)^n e^{-x^2}$$

a. Show $\left[\frac{d}{dx}, e^{-x^2/2} \right] = -x e^{-x^2/2}$.
b. Use this result to show

$$\begin{aligned} \left(-\frac{d}{dx} \right) e^{-x^2} &= e^{-x^2/2} \left(x - \frac{d}{dx} \right) e^{-x^2/2} \quad \text{and} \\ \left(-\frac{d}{dx} \right)^n e^{-x^2} &= e^{-x^2/2} \left(x - \frac{d}{dx} \right)^n e^{-x^2/2} \end{aligned}$$

c. As a result

$$H_n(x) e^{-x^2/2} = e^{+x^2/2} \left(-\frac{d}{dx} \right)^n e^{-x^2} = \left(x - \frac{d}{dx} \right)^n e^{-x^2/2}$$

d. Introduce the annihilation operator $a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})$, define the normalized ground state $\langle x|0\rangle$ by $a\langle x|0\rangle = 0$. Solve this equation, normalize the solution, and show $\langle x|0\rangle = e^{-x^2/2}/\sqrt{1\sqrt{\pi}}$.
e. Introduce the creation operator $a^\dagger = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$ and show

$$\langle x|n\rangle = \frac{(\sqrt{2}a^\dagger)^n}{\sqrt{2^n n!}} \langle x|0\rangle = \frac{H_n(x) e^{-x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}} = \psi_n(x) \qquad (6.33)$$

where $\psi_n(x)$ is the nth normalized harmonic oscillator eigenstate $\langle x|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\langle x|0\rangle$.

10. Assume a set of n harmonic oscillators interact through an angular momentum term ($L_{ij} = a_i^\dagger a_j - a_j^\dagger a_i$) and a quadrupole interaction ($Q_{ij} = a_i^\dagger a_j + a_j^\dagger a_i$).

a. Show that the hamiltonian for this system is

$$H = \sum_{i=1}^{n} \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2}\right) + i\sum_{i<j}\theta_{ij}\left(a_i^\dagger a_j - a_j^\dagger a_i\right) + \sum_{i\leq j} q_{ij}\left(a_i^\dagger a_j + a_j^\dagger a_i\right)$$

b. Show that this hamiltonian can be represented by a hermitian matrix. Show that for $i \leq j$the matrix elements are

$$\Gamma_{ij} = \hbar\omega_i\delta_{ij} + (q + i\theta)_{ij}$$

with $\Gamma^*_{ji} = \Gamma_{ij}$.

c. Show that an orthogonal transformation can be constructed so that the hamiltonian can be expressed in terms of n independent oscillators represented by creation and annihilation operators $b_i = m_{ij}a_j$: $H = \sum_{i=1}^{n} \hbar\omega_i'(b_i^\dagger b_i + \frac{1}{2}) + \text{cst.}$. Express the amplitudes m_{ij} in terms of the eigenvectors of $\Gamma(H)$.

d. Compute the shift in the zero point energy ("cst.").