5
Matrix Algebras

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The Lie algebras of the matrix Lie groups described in Chapter 3 are constructed. This is done by linearizing the constraints defining these matrix groups in the neighborhood of the identity operation.

5.1 Preliminaries

Lie algebras for the matrix groups treated in Chapter 3 are computed by linearizing the defining conditions in the neighborhood of the identity. The general linear groups $GL(n;F)$ have no defining condition (the only condition is $\det(M) \neq 0$), while examples (2)–(7) are already defined by linear constraints. Examples (8)–(11) are defined by bilinear and quadratic constraints that are linearized by applying the constraint to matrices infinitesimally close to the identity: $I + \epsilon A$. The matrices in

the Lie algebra are subject to easily derived linear constraints:

$$\begin{aligned} (I+\epsilon A)^\dagger G(I+\epsilon A) &= G \\ G+\epsilon(A^\dagger G+GA)+\mathcal{O}(\epsilon^2) &= G \\ A^\dagger G+GA &= 0 \end{aligned} \tag{5.1}$$

The special linear groups are defined by the n-linear constraint

$$\begin{aligned} \det(I+\epsilon A) &= 1+\epsilon\ \mathrm{Tr}(A)+\mathcal{O}(\epsilon^2)=1 \\ \mathrm{Tr}(A) &= 0 \end{aligned} \tag{5.2}$$

The matrix Lie algebras of the matrix Lie groups given in Chapter 3 are summarized below.

5.2 No Constraints

1. $\mathfrak{gl}(n;F)$. This algebra consists of arbitrary $n\times n$ matrices over the field F.

All remaining matrix algebras in this list are subalgebras of $\mathfrak{gl}(n;F)$.

5.3 Linear Constraints

The Lie algebras of the matrix groups have the same structure as the matrix groups. The only difference is that matrix elements that are constrained to be $+1$ in the groups are replaced by 0 in the algebra. All matrix algebras of matrix groups defined by linear constraints are summarized in Fig. 5.1.

2. $\mathfrak{ut}(p,q)$. **Upper Triangular** algebras. The matrix algebra has the same structure as the group $UT(p,q)$:

$$m_{i\alpha}=0 \qquad \begin{aligned} p+1 &\le i &&\le p+q \\ 1 &\le \alpha &&\le p \end{aligned} \tag{5.3}$$

3. $\mathfrak{ht}(p,q)$. The algebra for this class of groups is defined by the condition

$$m_{ij}=0 \qquad \begin{aligned} p+1 &\le i &&\le p+q \\ p+1 &\le j &&\le p+q \end{aligned} \tag{5.4}$$

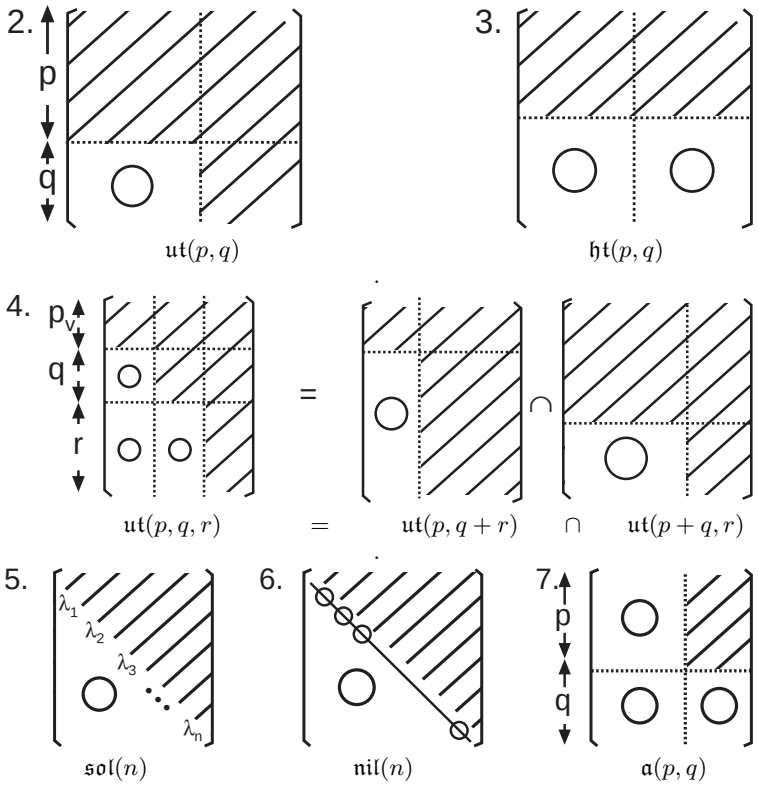

Fig. 5.1. Structure of the matrix algebras for groups defined by linear constraints.

Example: The group of affine transformations of the straight line consists of matrices

$$M(a,b) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \tag{5.5}$$

The identity is at $(a,b) = (1,0)$. Its algebra is spanned by the two operators

$$\left.\frac{\partial M}{\partial a}\right|_{(1,0)} = X_a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \left.\frac{\partial M}{\partial b}\right|_{(1,0)} = X_b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{5.6}$$

The commutation relations are given by $[X_a, X_b] = X_b$.

4. $\mathfrak{ut}(p,q,r)$. This matrix algebra is identical in structure to the parent group.

Example: A very useful 6-parameter subalgebra of $\mathfrak{ut}(1,2,1)$ is given by

$$\begin{bmatrix} 0 & l & r & -2\delta \\ 0 & \eta & 2R & -r \\ 0 & -2L & -\eta & l \\ 0 & 0 & 0 & 0 \end{bmatrix} = \eta X_\eta + R X_R + L X_L + r X_r + l X_l + \delta X_\delta \quad (5.7)$$

The commutation relations of the six infinitesimal generators of this matrix Lie algebra are summarized in the table below. The operator in the ith row and jth column is $[X_i, X_j]$.

	X_η	X_R	X_L	X_r	X_l	X_δ
X_η	0	$2X_R$	$-2X_L$	X_r	$-X_l$	0
X_R		0	$-4X_\eta$	0	$-2X_r$	0
X_L			0	$2X_l$	0	0
X_r				0	$-X_\delta$	0
X_l					0	0
X_δ						0

	$\hat{n}+\frac{1}{2}I$	$a^\dagger a^\dagger$	aa	$a^\dagger$	a	I
$\hat{n}+\frac{1}{2}I$	0	$2a^\dagger a^\dagger$	$-2aa$	$a^\dagger$	$-a$	0
$a^\dagger a^\dagger$		0	$-4(\hat{n}+\frac{1}{2}I)$	0	$-2a^\dagger$	0
aa			0	$2a$	0	0
$a^\dagger$				0	$-I$	0
a					0	0
I						0

(5.8)

The table inherits the antisymmetry of the commutator, so only one half has been presented. It is clear that there is an isomorphism between this matrix algebra and the algebra of the photon energy operator $\hat{n} = a^\dagger a + \frac{1}{2}$, two-photon creation and annihilation operators $a^\dagger a^\dagger$ and aa, single photon creation and annihilation operators $a^\dagger$ and a, and the identity operator $I = [a, a^\dagger]$. We observe that the 4×4 matrix X_δ representing the operator $I = [a, a^\dagger]$ is not diagonal. It needn't be, as long as it obeys the appropriate commutation relations.

5. $\mathfrak{sol}(n) = \mathfrak{ut}(1,1,1,\cdots,1)$. This matrix algebra is also identical in structure to its parent group. A very useful 4-parameter subalgebra of

$\mathfrak{ut}(1,1,1)$ is given by matrices of the following form

$$\begin{bmatrix} 0 & l & \delta \\ 0 & \eta & r \\ 0 & 0 & 0 \end{bmatrix} = \eta X_\eta + l X_l + r X_r + \delta X_\delta \tag{5.9}$$

The following commutation properties are easily verified

$$\begin{array}{rclrcl} [X_\eta, X_r] &=& +X_r & \left[a^\dagger a, a^\dagger\right] &=& +a^\dagger \\ [X_\eta, X_l] &=& -X_l & \left[a^\dagger a, a\right] &=& -a \\ [X_l, X_r] &=& X_\delta & \left[a, a^\dagger\right] &=& I \\ [X_\eta, X_\delta] &=& 0 & \left[a^\dagger a, I\right] &=& 0 \end{array} \tag{5.10}$$

6. $\mathfrak{nil}(n)$. Nilpotent matrices have an upper triangular structure, with $+1$ along the diagonal in the group and zeroes along the diagonal in the algebra. The three generators of the algebra of nilpotent 3×3 matrices have structure and commutation relations

$$\begin{bmatrix} 0 & l & \delta \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} = l X_l + r X_r + \delta X_\delta \tag{5.11}$$

$$\begin{array}{rclrcl} [X_l, X_r] &=& X_\delta & \left[a, a^\dagger\right] &=& I \\ [X_l, X_\delta] &=& 0 & [a, I] &=& 0 \\ [X_r, X_\delta] &=& 0 & \left[a^\dagger, I\right] &=& 0 \end{array} \tag{5.12}$$

These commutation relations are isomorphic to Heisenberg commutation relations. This is easily seen by setting $\eta = 0$ in (5.9). As a result, a number of difficult computations involving this algebra can be replaced by much simpler computations involving only 3×3 matrices.

7. $\mathfrak{a}(p,q)$. The matrix algebra for the commutative group of Example 7 in Chapter 3 [cf. (3.13)] consists of matrices having the form shown in Fig. 5.1

5.4 Bilinear and Quadratic Constraints

The nonlinear constraints that define the metric-preserving matrix Lie groups are easily converted to linear constraints that define their Lie algebras following the procedure described in (5.1) above.

8. Compact metric preserving groups. Matrices M for the algebras of these groups satisfy

$$M^\dagger + M = 0 \qquad \begin{array}{ccc} R & \mathfrak{o}(n) & \text{orthogonal} \\ C & \mathfrak{u}(n) & \text{unitary} \\ Q & \mathfrak{sp}(n) & \text{symplectic} \end{array} \tag{5.13}$$

The algebras of the orthogonal, unitary, and symplectic groups consist of $n \times n$ antihermitian matrices. The Lie algebras for the groups $O(3)$ and $U(2)$ are

$$\mathfrak{o}(3) = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} = \sum_i \theta_i L_i$$

$$\mathfrak{u}(2) = \frac{1}{2} \begin{bmatrix} ix_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & ix_0 - ix_3 \end{bmatrix} = \frac{i}{2} \sum_\mu x_\mu \sigma_\mu \tag{5.14}$$

The four 2×2 matrices σ_μ are called **Pauli spin matrices**.

9. Noncompact metric-preserving groups. Matrices M for the algebras of these groups satisfy

$$M^\dagger I_{p,q} + I_{p,q} M = 0 \qquad \begin{array}{ccc} R & \mathfrak{o}(p,q) \\ C & \mathfrak{u}(p,q) \\ Q & \mathfrak{sp}(p,q) \end{array} \tag{5.15}$$

The algebras of groups that leave invariant a nonsingular symmetric indefinite metric are most simply treated by determining their block diagonal structure. For example, the algebra $\mathfrak{so}(2,1)$ for the Lorentz group in the plane is

$$\begin{bmatrix} A^t & C^t \\ B^t & D \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} I_2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 0 \tag{5.16}$$

From this, we conclude

$$\begin{aligned} A &= -A^t \\ D &= -D = 0 \\ B^t &= C \end{aligned} \tag{5.17}$$

The matrix algebra of $\mathfrak{so}(2,1)$ is explicitly

$$\begin{bmatrix} 0 & \theta & v_1 \\ -\theta & 0 & v_2 \\ v_1 & v_2 & 0 \end{bmatrix} \tag{5.18}$$

By an identical argument the matrix Lie algebra for the Lorentz group $\mathfrak{so}(3,1)$ is

$$\begin{bmatrix} 0 & \theta_3 & -\theta_2 & v_1 \\ -\theta_3 & 0 & \theta_1 & v_2 \\ \theta_2 & -\theta_1 & 0 & v_3 \\ v_1 & v_2 & v_3 & 0 \end{bmatrix} \tag{5.19}$$

10. Antisymmetric nonsingular metric-preserving groups. Matrices M for the algebras of these groups satisfy

$$M^\dagger G + GM = 0 \qquad G^t = -G \qquad F = \begin{matrix} R & \mathfrak{sp}(G;R) \\ C & \mathfrak{sp}(G;C) \end{matrix} \tag{5.20}$$

Since G is nonsingular, $M = -G^{-1}M^\dagger G$ and

$$\text{Tr } M = -\text{Tr } G^{-1}M^\dagger G = -\text{Tr } M^\dagger = -\text{Tr } M^* = 0 \tag{5.21}$$

Therefore the trace of these matrices is imaginary.

11. Singular metric-preserving groups. Matrices M for the algebras of these groups satisfy

$$MG + GM^\dagger = 0 \qquad \begin{matrix} R & \mathfrak{o}(n;G) \\ C & \mathfrak{u}(n;G) \\ Q & \mathfrak{sp}(n;G) \end{matrix} \tag{5.22}$$

In the case that the $(p+q)\times(p+q)$ matrix G has singular block diagonal structure $\begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix}$ with $\det(g) \neq 0$, this constraint reduces to

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} Ag + gA^\dagger = 0 \\ C = C^\dagger = 0 \\ B, D \text{ arbitrary} \end{matrix} \tag{5.23}$$

In particular, in the case of real 4×4 matrices with singular symmetric metric $\text{diag}(1,1,1,0)$ the Lie algebra is

$$\begin{bmatrix} 0 & \theta_3 & -\theta_2 & t_1 \\ -\theta_3 & 0 & \theta_1 & t_2 \\ \theta_2 & -\theta_1 & 0 & t_3 \\ 0 & 0 & 0 & s_4 \end{bmatrix} \tag{5.24}$$

Here the parameters θ_i describe rotations about the ith coordinate axis and the t_i describe displacements of the origin along the ith coordinate direction. The parameter s_4 describes "scaling" of the time axis:

$t' = e^{s_4}t$. If s_4 is set to zero (traceless condition, see the following subsection) the Lie algebra is that of the Euclidean group $E(3) = ISO(3)$ (inhomogeneous rotation group in R^3).

The matrix Lie algebra for the Poincaré group $ISO(3,1)$ (3.24) is obtained by similar arguments using a singular 5×5 metric $G = \text{diag}\ (1,1,1,-1,0)$. The Lie algebra is (setting the trace equal to zero):

$$\begin{bmatrix} 0 & \theta_3 & -\theta_2 & v_1 & t_1 \\ -\theta_3 & 0 & \theta_1 & v_2 & t_2 \\ \theta_2 & -\theta_1 & 0 & v_3 & t_3 \\ v_1 & v_2 & v_3 & 0 & t_4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.25}$$

The Galilei group (3.25) has the following 5×5 matrix Lie algebra, obtained by "contraction" (cf., Chapter 13) from the Lie algebra of $ISO(3,1)$:

$$\begin{bmatrix} 0 & \theta_3 & -\theta_2 & v_1 & t_1 \\ -\theta_3 & 0 & \theta_1 & v_2 & t_2 \\ \theta_2 & -\theta_1 & 0 & v_3 & t_3 \\ 0 & 0 & 0 & 0 & t_4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.26}$$

5.5 Multilinear Constraints

12. Special Linear groups have algebras that satisfy the zero trace condition

$$\text{Tr } M = 0 \qquad F = \begin{matrix} R \\ C \\ Q \end{matrix} \quad \begin{matrix} \mathfrak{sl}(n,R) \\ \mathfrak{sl}(n,C) \\ \mathfrak{sl}(n,Q) \end{matrix} \tag{5.27}$$

The exponential of a matrix with zero trace is a matrix with determinant $+1$:

$$\det\left(e^M\right) = e^{\text{Tr}M} \tag{5.28}$$

5.6 Intersections of Groups

The Lie algebra for the intersection of two groups is the intersection of the groups' Lie algebras. The important algebra $\mathfrak{su}(n)$ is obtained from

the intersection of $\mathfrak{u}(n)$ and $\mathfrak{sl}(n;C)$ [cf. (5.14)]. For example

$$\mathfrak{su}(2) = \mathfrak{u}(2) \cap \mathfrak{sl}(2;C) = \frac{i}{2}\begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix} \tag{5.29}$$

5.7 Algebras of Embedded Groups

The Lie algebras of the embedded groups are constructed in a straightforward way.

The Lie algebra of $U(n)$ consists of $n \times n$ antihermitian matrices M:

$$M \in \mathfrak{u}(n) \Rightarrow (M^\dagger)_{ij} = -M^*_{ji} \tag{5.30}$$

The Lie algebra of $OU(2n)$ is obtained from the Lie algebra of $U(n)$ by replacing each of the $n(n-1)/2$ complex matrix elements M_{ij} $(i < j)$ $(M \in \mathfrak{u}(n))$ above the diagonal of M by a 2×2 real matrix, and each of the diagonal matrix elements M_{ii} by a real 2×2 matrix representing an imaginary complex number ($a = 0$, b arbitrary in Eq. (3.3)). The matrix elements M_{ij} below the diagonal $(i > j)$ are obtained from the antihermicity condition. The result is a real antisymmetric $2n \times 2n$ matrix with the property $\mathfrak{u}(n) \to \mathfrak{ou}(2n) \subset \mathfrak{o}(2n)$. The dimension of $\mathfrak{ou}(2n)$ is the dimension of $\mathfrak{u}(n)$: $2 \times n(n-1)/2 + 1 \times n = n^2$.

The Lie algebra of $Sp(n)$ consists of $n \times n$ antihermitian matrices M over Q:

$$M \in \mathfrak{sp}(n) \Rightarrow (M^\dagger)_{ij} = -M^*_{ji} \tag{5.31}$$

The adjoint is taken over the quaternion field. The Lie algebra of $USp(2n)$ is obtained from the Lie algebra of $Sp(n)$ by replacing each of the $n(n-1)/2$ quaternion matrix elements M_{ij} $(i < j)$ $(M \in \mathfrak{sp}(n))$ above the diagonal of M by a 2×2 complex matrix, and each of the diagonal matrix elements M_{ii} by a complex 2×2 matrix representing an imaginary quaternion ($q_0 = 0$, q_i arbitrary in Eq. (3.4)). The matrix elements M_{ij} below the diagonal $(i > j)$ are obtained from the antihermicity condition. The result is a real antihermitian $2n \times 2n$ matrix with the property $\mathfrak{sp}(n) \to \mathfrak{usp}(2n) \subset \mathfrak{su}(2n)$. The dimension of $\mathfrak{usp}(2n)$ is the dimension of $\mathfrak{sp}(n)$: $4 \times n(n-1)/2 + n \times 3 = 2n(2n+1)/2$.

5.8 Modular Groups

The modular group $GL(n;Z)$ has no Lie algebra because it is not a continuous group.

5.9 Basis Vectors

In each of these matrix algebras there is usually a clear choice of basis vectors. A useful choice is made by choosing a basis set that is orthogonal with respect to some inner product on the space of square matrices. In (5.6), (5.7), (5.9), (5.14), (5.18)-(5.19), and (5.24)-(5.26) the infinitesimal generators have been chosen to be orthogonal with respect to a convenient inner product.

As discussed in Chapter 4, Section 8, the Hilbert-Schmidt inner product on rectangular matrices

$$(X,Y) = \text{Tr } X^\dagger Y \tag{5.32}$$

is usually very useful. This inner product is positive-definite: $(X,X) = 0 \Rightarrow X = 0$. For example, for the algebra $\mathfrak{so}(2,1)$ (5.18), if X, X' are two 3×3 matrices in the algebra

$$(X',X) = \text{Tr } X'^\dagger X = 2(+\theta'\theta + v_1'v_1 + v_2'v_2) \tag{5.33}$$

There is a yet more useful inner product that can be defined on matrix Lie algebras. This is an analog of the Cartan-Killing inner product

$$(X,Y) = \text{Tr } XY \tag{5.34}$$

For $\mathfrak{so}(2,1)$ this inner product is

$$(X',X) = \text{Tr } X'X = 2(-\theta'\theta + v_1'v_1 + v_2'v_2) \tag{5.35}$$

This inner product is not positive-definite. For giving up positive-definiteness we gain information of both an algebraic and a topological nature. At the algebraic level, the subspace on which this inner product is identically zero is the largest nilpotent invariant subalgebra (subalgebra of matrices equivalent to upper-triangular matrices) in the original algebra. The subspace on which the inner product is negative definite consists of compact group generators, and the subspace on which it is positive definite consists of noncompact generators. When appropriate measures are taken (in Chapter 11), the negative definite subspace closes under commutation, so describes a compact Lie group.

The Cartan-Killing inner product is defined in terms of the structure

constants of a Lie algebra. These are incorporated into the regular matrix representation of the Lie algebra. The Cartan-Killing inner product (X, Y) is specifically defined as the trace of the product of the regular matrix representatives of X and Y. Other inner products are conveniently defined when other matrix representations are used. In many instances it is very convenient to use the defining matrix representation of the Lie algebra: this representation certainly contains no less information than the regular matrix representation. For a large class of Lie algebras (simple Lie algebras) these two different inner products are strictly proportional.

It is remarkable that this metric contains information of both a topological and an algebraic nature. To illustrate the difference between the compact and noncompact cases, we consider 2×2 matrices

$$\begin{array}{lllllll} X & = & \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} & (X,X) & = & -2 & e^{\theta X} & = & \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ Y & = & \begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix} & (Y,Y) & = & +2 & e^{\theta X} & = & \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \end{array} \tag{5.36}$$

In the compact case, the group element $\exp(\theta X)$ periodically returns to the identity as θ increases. Therefore the group can be parameterized by a finite range of parameter values: $-\pi \leq \theta \leq +\pi$, with $-\pi$ and $+\pi$ identified. On the other hand, in the noncompact case the group is parameterized by the entire line $-\infty < \theta < +\infty$. The underlying manifolds for the two groups are the circle S^1 and the line R^1.

In the compact case the simplification of parameterizing the group with a bounded subset of the Lie algebra ($-\pi \leq \theta \leq +\pi$) is somewhat offset by the complication of matching boundary conditions — identifying the group operations parameterized by $-\pi$ and $+\pi$. In the noncompact case the simplification of not having to worry about matching boundary conditions is somewhat offset by the fact that it takes the entire subspace in the Lie algebra, R^k, where k is the number of noncompact generators, to parameterize this piece of the group. This piece of the group is topologically identical to R^k – that it, it is Euclidean. These remarks will be clarified and elaborated on in Chapter 7.

5.10 Conclusion

In this chapter we have constructed the Lie algebras for all the matrix Lie groups defined in Chapter 3. This is done by linearizing the constraints that define the original matrix Lie groups in the neighborhood of the identity. For the general linear groups which are defined by no constraints, the Lie algebras $\mathfrak{gl}(n;F)$ are also defined by no constraints. For the Lie groups defined by linear constraints, linearization is trivial and produces a matrix Lie algebra having structure identical to that of the parent Lie group. Transition from the Lie group to the Lie algebra replaces nonlinear constraints by linear conditions defining the Lie algebras of the metric preserving groups ($G = I_n, I_{p,q}$, nonsingular antisymmetric, general nonsingular) and the unimodular groups. One natural way to choose basis vectors in these Lie algebras has been described.

5.11 Problems

1. The Lie group $UT(1,1)$ has Lie algebra of the form

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = aX_a + bX_b + cX_c$$

Show that in this matrix Lie algebra an inner product can be defined by $(A,A) = \text{tr } (A)^2 = a^2 + c^2$.

2. Show that the regular representation of the matrix Lie algebra $\mathfrak{ut}(1,1)$ given in Problem 1 is

$$R(A) = \left[\begin{array}{cc|c} 0 & 0 & -b \\ 0 & 0 & +b \\ \hline 0 & 0 & a-c \end{array}\right] \begin{array}{c} X_a \\ X_c \\ X_b \end{array}$$

with the ordering of the basis vectors given on the right. Show that the Cartan-Killing inner product in the regular representation is $(A,A) = \text{tr } R(A)^2 = (a-c)^2$. The inner product in the regular representation suggests that the linear combination $X_a + X_c$ commutes with all operators in the Lie algebra. Is this true?

3. Write down the algebra inclusions $\mathfrak{gl}(1;R) \subset \mathfrak{gl}(1;C) \subset \mathfrak{gl}(1;Q)$ explicitly in terms of the 2×2 complex matrices as defined in (3.3) and (3.4).

4. Construct the table [analogous to (5.8)] giving the commutation

relations for the photon energy operator $\hat{n} = a^\dagger a + \frac{1}{2}$, creation and annihilation operators $a^\dagger$ and a, and the identity operator I. Compare with a table for the commutation relations of the matrices $X_\eta, X_r, X_l, X_\delta$ defined in (5.9). Show that the two Lie algebras are isomorphic. The photon number operator is $\hat{n} = a^\dagger a$ and the photon energy operator is $\hat{E} = (a^\dagger a + \frac{1}{2})\hbar\omega \to a^\dagger a + \frac{1}{2}$ for $\hbar\omega = 1$.

5. **Cartan decomposition:** Assume a matrix Lie algebra has a block diagonal structure given by

$$Z = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right] + \left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right]$$

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$$

Show that this decomposition satisfies the commutation relations

$$\begin{array}{ll} [\mathfrak{h},\mathfrak{h}] & \subseteq \mathfrak{h} \\ [\mathfrak{h},\mathfrak{p}] & \subseteq \mathfrak{p} \\ [\mathfrak{p},\mathfrak{p}] & \subseteq \mathfrak{h} \end{array}$$

This means that if $X, X' \in \mathfrak{h}$ and $Y, Y' \in \mathfrak{p}$, then $[X, X'] \in \mathfrak{h}$, $[X, Y] \in \mathfrak{p}$, $[Y, Y'] \in \mathfrak{h}$. Conclude that the subspace $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Is $\mathfrak{p}$ a subalgebra (under what conditions is $\mathfrak{p}$ a subalgebra)?

6. Show that an inner product for the Cartan decomposition given in the previous problem is

$$(Z, Z) = \text{Tr } Z^2 = \text{Tr } A^2 + \text{Tr } BC + \text{Tr } CB + \text{Tr } D^2$$

If $X = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathfrak{h}$ and $Y = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathfrak{p}$, then

$$(X, X) = \text{Tr } (A^2 + D^2) \qquad (Y, Y) = \text{Tr } (BC + CB)$$

Show that X and Y are orthogonal under this inner product: $(X, Y) = 0$.

7. The Lie algebra $\mathfrak{so}(p, q)$ has the structure $\begin{bmatrix} A & B \\ B^t & C \end{bmatrix}$ where the $p \times p$ and $q \times q$ matrices A and C satisfy $A^t = -A$ and $C^t = -C$. If $X = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \in \mathfrak{h}$ and $Y = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \in \mathfrak{p}$, show

- $(X, Y) = 0$
- $(X, X) \leq 0, \quad (X, X) = 0 \Rightarrow X = 0$
- $(Y, Y) \geq 0, \quad (Y, Y) = 0 \Rightarrow Y = 0$

These results are summarized by

$$\begin{array}{rcl}(\mathfrak{h},\mathfrak{h}) & \leq & 0\\ (\mathfrak{h},\mathfrak{p}) & = & 0\\ (\mathfrak{p},\mathfrak{p}) & \geq & 0\end{array}$$

8. The Lie algebra $\mathfrak{su}(p,q)$ has the structure $\begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix}$ where the $p\times p$ and $q\times q$ matrices A and C satisfy $A^\dagger = -A$, $C^\dagger = -C$, and Tr $(A+C)=0$. If $X = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \in \mathfrak{h}$ and $Y = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \in \mathfrak{p}$, then show once again that

$$\begin{array}{rcl}(\mathfrak{h},\mathfrak{h}) & \leq & 0\\ (\mathfrak{h},\mathfrak{p}) & = & 0\\ (\mathfrak{p},\mathfrak{p}) & \geq & 0\end{array}$$

Show that $(X,X)=0 \Rightarrow X=0$ and $(Y,Y)=0 \Rightarrow Y=0$.

9. The Lie algebra for $\mathfrak{sl}(n;R)$ has a decomposition in terms of real antisymmetric and traceless symmetric matrices $A^t=-A$ and $B^t=B$ with Tr $B=0$:

$$\begin{array}{rcccc}\mathfrak{sl}(n;R) & = & A & + & B\\ \mathfrak{g} & = & \mathfrak{h} & + & \mathfrak{p}\end{array}$$

Show

$$\begin{array}{rcl}[\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h}\\ [\mathfrak{h},\mathfrak{p}] & \subseteq & \mathfrak{p}\\ [\mathfrak{p},\mathfrak{p}] & \subseteq & \mathfrak{h}\end{array} \quad \text{and} \quad \begin{array}{rcl}(\mathfrak{h},\mathfrak{h}) & \leq & 0\\ (\mathfrak{h},\mathfrak{p}) & = & 0\\ (\mathfrak{p},\mathfrak{p}) & \geq & 0\end{array}$$

Show that $(A,A)=0 \Rightarrow A=0$ and $(B,B)=0 \Rightarrow B=0$.

10. The Lie algebra for $\mathfrak{sl}(n;C)$ has a decomposition in terms of traceless antihermitian matrices $A^\dagger=-A$ and traceless hermitian matrices $H^\dagger = H$:

$$\begin{array}{rcccc}\mathfrak{sl}(n;C) & = & \text{Antihermitian} & + & \text{Hermitian}\\ \mathfrak{g} & = & \mathfrak{h} & + & \mathfrak{p}\end{array}$$

Show

$$\begin{array}{rcl}[\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h}\\ [\mathfrak{h},\mathfrak{p}] & \subseteq & \mathfrak{p}\\ [\mathfrak{p},\mathfrak{p}] & \subseteq & \mathfrak{h}\end{array} \quad \text{and} \quad \begin{array}{rcl}(\mathfrak{h},\mathfrak{h}) & \leq & 0\\ (\mathfrak{h},\mathfrak{p}) & = & 0\\ (\mathfrak{p},\mathfrak{p}) & \geq & 0\end{array}$$

Show that $(A,A)=0 \Rightarrow A=0$ and $(H,H)=0 \Rightarrow H=0$.

11. Assume that $\mathfrak{g}$ is a Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$, with commutation relations and inner product properties given by

$$\begin{array}{rcl} [\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h} \\ [\mathfrak{h},\mathfrak{p}] & \subseteq & \mathfrak{p} \\ [\mathfrak{p},\mathfrak{p}] & \subseteq & \mathfrak{h} \end{array} \quad \text{and} \quad \begin{array}{rcl} (\mathfrak{h},\mathfrak{h}) & \leq & 0 \\ (\mathfrak{h},\mathfrak{p}) & = & 0 \\ (\mathfrak{p},\mathfrak{p}) & \geq & 0 \end{array}$$

Show that if every $n \times n$ matrix B in $\mathfrak{p}$ is multiplied by i and the resulting algebra is defined by $\mathfrak{g}' = \mathfrak{h} + i\mathfrak{p} = \mathfrak{h} + \mathfrak{p}'$ then

$$\begin{array}{rcl} [\mathfrak{h},\mathfrak{h}] & \subseteq & \mathfrak{h} \\ [\mathfrak{h},\mathfrak{p}'] & \subseteq & \mathfrak{p}' \\ [\mathfrak{p}',\mathfrak{p}'] & \subseteq & \mathfrak{h} \end{array} \quad \text{and} \quad \begin{array}{rcl} (\mathfrak{h},\mathfrak{h}) & \leq & 0 \\ (\mathfrak{h},\mathfrak{p}') & = & 0 \\ (\mathfrak{p}',\mathfrak{p}') & \leq & 0 \end{array}$$

In short, noncompact algebras that satisfy a Cartan decomposition can be analytically continued to compact algebras.

12. Extend the Cartan decomposition and analytic continuation arguments to the quaternion algebra $\mathfrak{g} = \mathfrak{sl}(n;Q)$ with respect to the subalgebra $\mathfrak{h} = \mathfrak{sl}(n;C)$.

13. A matrix Lie algebra has the form

$$A = \left[\begin{array}{ccc|c|c} 0 & \theta_3 & -\theta_2 & b_1 & t_1 \\ -\theta_3 & 0 & \theta_1 & b_2 & t_2 \\ \theta_2 & -\theta_1 & 0 & b_3 & t_3 \\ \hline \mu b_1 & \mu b_2 & \mu b_3 & 0 & t_4 \\ \hline \sigma t_1 & \sigma t_2 & \sigma t_3 & -\mu\sigma t_4 & 0 \end{array}\right] \qquad \begin{array}{rc} \frac{1}{2}(A,A) = & -(\theta\cdot\theta) \\ & +\mu(\mathbf{b}\cdot\mathbf{b}) \\ & +\sigma(\mathbf{t}\cdot\mathbf{t}) \\ & -\mu\sigma t_4^2 \end{array}$$

Show

μ	σ	Algebra	Singular Subspace
$+1$	$+1$	$\mathfrak{so}(3,2)$	
-1	$+1$	$\mathfrak{so}(4,1)$	
-1	-1	$\mathfrak{so}(5)$	
$+1$	0	Poincare	Translations t_μ
0	0	Galilei	Translations t_μ, Boosts $\mathbf{b}$

14. Assume that $\mathfrak{g} = A$, where A is a Lie algebra of $n \times n$ matrices on which the inner product is negative-definite: Tr $A^2 \leq 0, = 0 \Rightarrow A = 0$. Then show that $EXP(tA)$ returns to any neighborhood of the identity I_n if t becomes large enough. If the eigenvalues of A are rationally related

($\lambda_i = \gamma n_i$, n_i are integers, $\gamma \neq 0$ is rational or irrational), $EXP(tA)$ returns periodically to I_n. What is t_0, the minimum period in t?

15. Use the parameterization of $\mathfrak{so}(3)$ given in Problem 3.14. Show that the differentials (dx, dy, dz) of a point in the neighborhood of (x, y, z) are related to the displacements $(\delta x, \delta y, \delta z)$ in the neighborhood of the identity by

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} m_{11} & 0 & -y \\ 0 & m_{11} & x \\ 0 & m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$$

Use the values you constructed for the matrix elements m_{ij} to construct explicitly the metric tensor $g(x, y, z)$ and the invariant measure $d\mu(x, y, z)$ on the group $SO(3)$ with this parameterization.

16. $\mathfrak{g}$ is a matrix Lie algebra. Show that if the matrix subspaces $\mathfrak{h}$ and $\mathfrak{p}$ defined below exist in the algebra ($\mathfrak{g} \cap \mathfrak{h} = \mathfrak{h}$, $\mathfrak{g} \cap \mathfrak{p} = \mathfrak{p}$),

	$\mathfrak{h}$	$\mathfrak{p}$
(a)	$\mathfrak{g} + \mathfrak{g}^*$	$\mathfrak{g} - \mathfrak{g}^*$
(b)	$\mathfrak{g} - \mathfrak{g}^t$	$\mathfrak{g} + \mathfrak{g}^t$
(c)	$\mathfrak{g} - \mathfrak{g}^\dagger$	$\mathfrak{g} + \mathfrak{g}^\dagger$

then the following commutation relations are satisfied:

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \qquad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p} \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$$