3

Matrix Groups

Contents

Almost all Lie groups encountered in the physical sciences are matrix groups. In this chapter we describe most of the matrix groups that are typically encountered. These include the general linear groups $GL(n;F)$ of nonsingular $n \times n$ matrices over the fields F of real numbers, complex numbers, and quaternions, and various of their subgroups obtained by imposing linear, bilinear and quadratic, and n-linear constraints on these matrix groups.

3.1 Preliminaries

It is first useful to state a simple theorem.

Definition: A subgroup H of G (also $H \subset G$) is a subset of G that is also a group under the group multiplication of G.

Example: The set of matrices

$$\begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} \tag{3.1}$$

is a subgroup of $SL(2;R)$.

Theorem: If $H_1 \subset G$ and $H_2 \subset G$ are subgroups of G then their intersection $H_{12} = H_1 \cap H_2$ is a subgroup of G.

Proof: Verify that the four group axioms are satisfied for all operations in $H_1 \cap H_2$.

Example: If H_1 is the 2-dimensional subgroup of $SL(2; R)$ described in (3.1) above and H_2 is the one-dimensional subgroup of 2×2 orthogonal matrices

$$H_2 = SO(2) = \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \right] \qquad \theta \in [0, 2\pi) \tag{3.2}$$

then the intersection $H_1 \cap H_2$ is the zero dimensional subgroup containing the two discrete group operations $\pm I_2$.

The matrix groups that we consider are defined over the fields of real numbers ($F = R$), complex numbers ($F = C$), and quaternions ($F = Q$). The complex numbers can be constructed from pairs of real numbers by adjoining a square root of -1. Their multiplication properties can be analyzed by mapping the pair of real numbers into 2×2 matrices

$$c = (a, b) = a + ib \qquad a \in R, \quad b \in R, \quad i^2 = -1$$

$$(a, b) \longrightarrow \left[\begin{array}{cc} a & b \\ -b & a \end{array} \right] \qquad i = (0, 1) \longrightarrow \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \tag{3.3}$$

In an analogous way, the quaternions can be constructed from pairs of complex numbers by adjoining another square root of -1, and their multiplication properties analyzed by mapping the pair of complex numbers into 2×2 matrices

$$q = (c_1, c_2) = c_1 + jc_2 \qquad \begin{array}{l} c_1 = a_1 + ib_1 \in C \\ c_2 = a_2 + ib_2 \in C \\ i^2 = -1, \quad j^2 = -1, \quad ij + ji = 0 \end{array}$$

$$(c_1, c_2) \longrightarrow \left[\begin{array}{cc} c_1 & c_2 \\ -c_2^* & c_1^* \end{array} \right] \tag{3.4}$$

The mapping of two complex numbers into a 2×2 matrix representing a quaterion can also be expressed as a mapping of four real numbers into a 2×2 matrix representing a quaterion:

$$q_0 + q_1\mathcal{I} + q_2\mathcal{J} + q_3\mathcal{K} \rightarrow \left[\begin{array}{cc} q_0 + iq_3 & iq_1 + q_2 \\ iq_1 - q_2 & q_0 - iq_3 \end{array} \right]$$

The four basis vectors $1, \mathcal{I}, \mathcal{J}, \mathcal{K}$ for this map are related to the four

Pauli spin matrices, and i is the usual square root of -1 introduced above in Eq. (3.3). The details are presented in Problem # 1 at the end of this chapter.

We list, in order, matrix groups on which no constraints are imposed (1), on which only linear constraints are imposed (2)–(7), on which bilinear and quadratic constraints are imposed (8)–(11), and on which n-linear or multilinear constraints [det(M) = +1] are imposed (12).

3.2 No Constraints

1. $GL(n;F)$. **General Linear** groups consist of nonsingular $n \times n$ matrices over the real, complex, or quaternion fields. The group $GL(1;Q)$ consists of 1×1 quaternion, or 2×2 complex matrices that satisfy

$$\det \left[\begin{array}{cc} a_1 + ib_1 & a_2 + ib_2 \\ -a_2 + ib_2 & a_1 - ib_1 \end{array} \right] = a_1^2 + b_1^2 + a_2^2 + b_2^2 \neq 0 \tag{3.5}$$

The determinant of an $n \times n$ matrix A with matrix elements A_i^j is defined by

$$\det\ (A) = \sum_I \sum_J \frac{1}{n!}\ \epsilon^{i_1 i_2 \ldots i_n}\ A_{i_1}^{j_1} A_{i_2}^{j_2} \ldots A_{i_n}^{j_n}\ \epsilon_{j_1 j_2 \ldots j_n}$$

Here $\epsilon^{i_1 i_2 \cdots i_n}$ and its covariant version are the Levi-Civita symbols: $+1$ for an even permutation of the integers $1, 2, \ldots, n$; -1 for an odd permutation; and 0 if two or more values of the indices i_* are equal. With this definition there is no difficulty computing the determinant of a matrix containing matrix elements that do not commute (quaternions).

All remaining matrix groups in this list are subgroups of $GL(n;F)$.

3.3 Linear Constraints

These matrix groups all have a block structure or an echelon block structure. The linear constraints simply require specific blocks of matrix elements to vanish, or require some diagonal matrix elements to be $+1$. The structures of all these matrix groups are exhibited in Fig. 3.1.

2. $UT(p,q)$. **Upper Triangular** groups. The $n \times n$ $(n = p+q)$ matrix is partitioned into block form and an off-diagonal block is constrained to be zero

$$m_{i\alpha} = 0 \qquad \begin{array}{ccccc} p+1 & \leq & i & \leq & p+q \\ 1 & \leq & \alpha & \leq & p \end{array} \tag{3.6}$$

Example: The action of transformations in $UT(1,1)$ on the plane R^2 is as follows:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ dy \end{bmatrix} \tag{3.7}$$

The x-axis $y = 0$ remains invariant. It is an invariant subspace ($y = 0 \rightarrow y' = 0$), mapped into itself by all group operations in $UT(1,1)$. The y-axis $x = 0$ is not invariant. More generally, if $UT(p,q)$ acts on the direct sum vector space $V_p \oplus V_q$, the subspace V_q is invariant while V_p is not. For lower triangular matrices reverse p and q.

3. $HT(p,q)$. This is a subgroup of $UT(p,q)$ obtained by imposing the additional linear constraints on the matrix elements of a diagonal block

$$m_{ij} - \delta_{ij} = 0 \qquad \begin{array}{ccccc} p+1 & \leq & i & \leq & p+q \\ p+1 & \leq & j & \leq & p+q \end{array} \tag{3.8}$$

Example: Affine transformations in $HT(1,1)$ ($m_{22} = 1$) act on the x-axis by $x \rightarrow x' = ax + b$:

$$\begin{bmatrix} x' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} ax+b \\ 1 \end{bmatrix} \tag{3.9}$$

4. $UT(p,q,r)$. This matrix group consists of upper triangular matrices that are the intersection of the matrix groups $UT(p,q+r) \cap UT(p+q,r)$.

Example: We consider 4×4 complex matrices with the structure

$$\left[\begin{array}{c|cc|c} 1 & \star & \star & \star \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{SU(1,1)}} & \star \\ 0 & & & \star \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{3.10}$$

where the 2×2 matrix $SU(1,1)$ is defined below in (3.30). Matrix groups with the structure (3.10) are encountered in treatments of squeezed states of the electromagnetic field and scattering of projectiles from simple diatomic molecules [39, 40].

5. $Sol(n) = UT(1,1,1,\cdots,1)$. **Solvable** groups are strictly upper triangular.

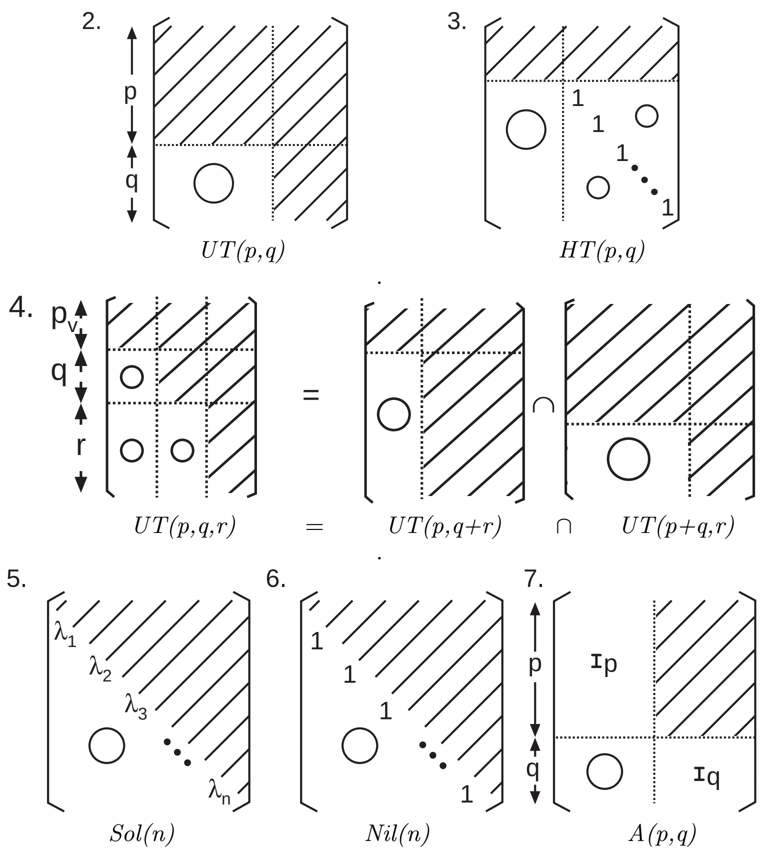

Fig. 3.1. Structure of the matrix groups defined by linear constraints.

Example: We consider the subgroup of 3×3 matrices in $UT(1,1,1)$ of the form

$$\begin{bmatrix} 1 & l & d \\ 0 & \eta & r \\ 0 & 0 & 1 \end{bmatrix} \tag{3.11}$$

These matrices have the same structure as the group generated by exponentials of the photon number operator ($\hat{n} = a^\dagger a$), the creation ($a^\dagger$) and annihilation (a) operators, and their commutator ($I = aa^\dagger - a^\dagger a = [a, a^\dagger]$). We will use this identification between operator and matrix groups to develop some powerful operator disentangling theorems.

6. $Nil(n)$. **Nilpotent** groups are subgroups of $Sol(n)$ whose diagonal matrix elements are all $+1$.

Example: Matrices in $Nil(3)$ of the form

$$\begin{bmatrix} 1 & l & d \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} \tag{3.12}$$

are closely related to the photon creation and annihilation operators $(a^\dagger, a, I)$ and the group generated by the exponentials of the position and momentum operators (p and q) and their commutator $[p, q] = \frac{\hbar}{i}$. This 3×3 matrix group is called the Heisenberg group. (It is technically the covering group of the Heisenberg group.) The set of change of basis transformations $\langle p|q \rangle = \frac{1}{\sqrt{2}} e^{2\pi i p q/h}$ encountered in Quantum Mechanics is a unitary representation of this group of 3×3 matrices.

7. $A(p, q)$. This group consists of matrices that are the sum of an identity matrix and the upper right hand off-diagonal block of a (p, q) blocked matrix. Its matrix elements satisfy

$$\begin{array}{rcll} A_{i,j} & = & \delta_{i,j} & 1 \leq i, j \leq p \\ A_{\alpha,\beta} & = & \delta_{\alpha,\beta} & p+1 \leq \alpha, \beta \leq p+q \\ A_{\alpha,j} & = & 0 & \\ A_{i,\beta} & = & \text{arbitrary} & \end{array}$$

This groups is abelian or commutative: $AB = BA$ for all elements (matrices) in this group.

Example: We consider the translation subgroup $A(1, 1)$ of the affine group of transformations of the x-axis (3.9): $x \rightarrow x' = x + a$. Successive transformations of this type commute

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \tag{3.13}$$

For $A(p, q)$, p and q are $p \times q$ matrices.

3.4 Bilinear and Quadratic Constraints

In (8)–(11) we treat groups that preserve a metric, represented by a matrix G. They all satisfy the bilinear or quadratic constraint condition $M^\dagger G M = G$. If G is symmetric positive-definite we can set $G = I_n$ (8). If G is nonsingular and symmetric but indefinite we can set $G = I_{p,q}$ (9). If G is nonsingular and antisymmetric, we can take $G = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$

(10). These are the groups that leave Hamilton's equations of motion invariant in form. A large spectrum of interesting groups occurs if G is singular (11). The matrix elements in these cases are defined by both bilinear *and* linear conditions.

8. Compact metric preserving groups. Matrices M in these groups satisfy the quadratic condition $M^{\dagger}GM = G$, where G is symmetric positive-definite, and which we can take as I_n

$$G = I_n \qquad \begin{array}{ccl} R & O(n) & \text{Orthogonal Group} \\ C & U(n) & \text{Unitary Group} \\ Q & Sp(n) & \text{Symplectic Group} \end{array} \tag{3.14}$$

These are groups of rotations that leave invariant a positive-definite metric in a real, complex, or quaternion valued n-dimensional linear vector space. The manifolds that parameterize these groups are compact because the condition $M^{\dagger}GM = G$ defines matrices that form closed bounded subsets of the manifolds that parameterize the matrix groups $GL(n; F)$, $F = R, C, Q$.

Example: As examples we introduce real 3×3 matrices of rigid rotations (and inversions) in R^3, complex 2×2 matrices that preserve inner products in a complex two-dimensional linear vector space C^2 (of spin states, for example), and quaternion valued 1×1 matrices that preserve length in a one-dimensional linear vector space over Q

$$\begin{array}{ccccccccc} M^{\dagger}I_3M & = & I_3 & M & \in & O(3) & F & = & R \\ M^{\dagger}I_2M & = & I_2 & M & \in & U(2) & F & = & C \\ M^{\dagger}I_1M & = & I_1 & M & \in & Sp(1) & F & = & Q \end{array} \tag{3.15}$$

The group $SU(1; Q)$ is the subgroup of $GL(1; Q)$ (3.5) subject to the condition

$$a_1^2 + b_1^2 + a_2^2 + b_2^2 = 1 \tag{3.16}$$

This group is geometrically equivalent to the 3-dimensional sphere embedded in R^4

$$SL(1; Q) \sim S^3 \subset R^4 \tag{3.17}$$

We will see many other relations between groups and geometry.

9. Noncompact metric-preserving groups. Matrices in these groups leave invariant a nonsingular symmetric but indefinite metric G, which we take as $G = I_{p,q}$, $p + q = n$. This is a diagonal matrix with p elements $+1$ and q elements -1 along the diagonal. Matrices M in these groups satisfy the quadratic condition $M^{\dagger}GM = G$, where

$$G = I_{p,q} \qquad \begin{array}{lll} R & O(p,q) & \text{Orthogonal Group} \\ C & U(p,q) & \text{Unitary Group} \\ Q & Sp(p,q) & \text{Symplectic Group} \end{array} \tag{3.18}$$

The manifolds that parameterize these groups are noncompact when $p \neq 0$, $q \neq 0$. These noncompact groups are related by analytic continuation to corresponding compact metric-preserving groups.

Example: The Lorentz group preserves the invariant $x^2 + y^2 + z^2 - (ct)^2$ and is thus defined by the condition

$$M^t I_{3,1} M = I_{3,1}$$

$$\begin{gathered} \begin{bmatrix} A^t & C^t \\ B^t & D^t \end{bmatrix} \begin{bmatrix} I_3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \\ \begin{bmatrix} A^tA - C^tC & A^tB - C^tD \\ B^tA - D^tC & B^tB - D^tD \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & -1 \end{bmatrix} \end{gathered} \tag{3.19}$$

There are much better ways to parameterize this group. These involve exponentiating its Lie algebra.

10. Antisymmetric metric-preserving groups. The metric G is an $N \times N$ nonsingular antisymmetric matrix

$$M^t G M = G \qquad F = \begin{array}{ll} R & Sp(N,R) \\ C & Sp(N,C) \end{array} \tag{3.20}$$

Since $\det(G) = \det(G^t) = \det(-G) = (-)^N \det(G)$, N must be even: $N = 2n$. The metric matrix can be chosen to have the canonical forms $G = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ or $G = \sum_{\alpha=1}^{n} \oplus \left[i\sigma_y\right]_{\alpha}$. This consists of n copies of the matrix $i\sigma_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ along the diagonal. Symplectic transformations in $Sp(2n;R)$ leave invariant the form of the classical hamiltonian equations of motion.

Example: The symplectic group $Sp(2;R) \subset GL(2;R)$ satisfies the constraint

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & ad - bc \\ bc - ad & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.21}$$

The constraint is $ad - bc = +1$. Thus, $Sp(2;R) = SL(2;R)$.

11. General metric-preserving groups. Matrices in these groups leave invariant a singular metric G.

$$\begin{array}{cc} R & O(n;G) \\ C & U(n;G) \\ Q & Sp(n;G) \end{array} \tag{3.22}$$

Example: We consider 4×4 real matrices and choose

$$G = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.23}$$

Partitioning M into blocks and imposing the condition $MGM^t = G$, we find

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^t & C^t \\ B^t & D \end{bmatrix} = \begin{bmatrix} AA^t & AC^t \\ CA^t & CC^t \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.24}$$

This results in the conditions

$$\begin{array}{llll} AA^t & = & I_3 & \text{quadratic constraints, } A \in O(3) \\ C & = & 0 & \text{linear constraints} \\ B, D & & \text{arbitrary} & \text{no constraints} \end{array}$$

The subgroup obtained by setting the 1×1 submatrix D equal to $+1$ is the Euclidean group $E(3)$ whose action on the coordinates (x, y, z) of a point in R^3 is

$$\left[\begin{array}{c} x' \\ y' \\ z' \\ \hline 1 \end{array}\right] = \left[\begin{array}{ccc|c} & & & t_1 \\ & A & & t_2 \\ & & & t_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \\ z \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} + & t_1 \\ + & t_2 \\ + & t_3 \end{matrix} \\ \hline 1 \end{array}\right] \tag{3.25}$$

That is, the coordinates are rotated by the matrix A and translated by the vector $\mathbf{t}$. By closely similar arguments the Poincaré group, consisting of Lorentz transformations $[A \in SO(3,1),\ AI_{3,1}A^t = I_{3,1}\ (3.17)]$ and space-time displacements is isomorphic to the real 5×5 matrix group

$$\text{Poincaré group :} \qquad \left[\begin{array}{c|c} O(3,1) & t \\ \hline 0 & 1 \end{array}\right] \tag{3.26}$$

The Galilei group consists of rotations in R^3, transformations to a coordinate system moving with velocity $\mathbf{v}$, and displacements of space $(\mathbf{t})$

and time (t_4) coordinates. It is isomorphic to the group of 5×5 matrices with the structure

$$\text{Galilei group}: \quad \left[\begin{array}{c|c|c} O(3) & \mathbf{v} & \mathbf{t} \\ \hline 0 & 1 & t_4 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{3.27}$$

3.5 Multilinear Constraints

It is possible to impose trilinear, 4-linear, ..., constraints on $n \times n$ matrices. This requires a great deal of effort, and leads to few results, principle among which are the five exceptional Lie groups that we will meet in Chapter 10. The only multilinear constraint that leads systematically to a large class of Lie groups is the n-linear constraint, defined by the determinant.

12. Special Linear Groups, or **Unimodular Groups**. These are defined by the condition

$$\det M = +1 \qquad F = \begin{array}{ll} R & SL(n,R) \\ C & SL(n,C) \\ Q & SL(n,Q) \end{array} \tag{3.28}$$

Example: The group $SL(2;R)$ has previously been encountered. The subset of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2;R) \subset GL(2;R)$ satisfies the constraint $ad - bc = +1$, which is bilinear.

3.6 Intersections of Groups

Some important groups are intersections of those listed above

$$\begin{array}{lcrcl} SO(n) & = & O(n) & \cap & SL(n;R) \\ SO(p,q) & = & O(p,q) & \cap & SL(p+q;R) \\ SU(n) & = & U(n) & \cap & SL(n;C) \\ SU(p,q) & = & U(p,q) & \cap & SL(p+q;C) \end{array} \tag{3.29}$$

Example: We construct the three-dimensional noncompact group $SU(1,1)$ by taking the intersection of $U(1,1)$ with $SL(2;C)$:

$$SU(1,1) = U(1,1) \cap SL(2;C) \rightarrow \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix} \tag{3.30}$$

where $a^*a - b^*b = +1$.

3.7 Embedded Groups

The unitary group $U(n)$ consists of $n \times n$ complex matrices that obey the constraint $U^\dagger U = I_n$. For some purposes it is useful to represent this group as a group of real matrices. This is done by replacing each of the complex entries in $U(n)$ by a real 2×2 matrix according to the prescription given in Eq. (3.3). The resulting matrix is a real $2n \times 2n$ matrix M. This matrix inherits the constraint that comes with the unitary group, $U^\dagger U = I_n$. This constraint now appears in the form $M^t M = I_{2n}$. We have been able to replace † by ^{t} since the matrices are real, and must replace I_n by I_{2n} since the matrices are $2n \times 2n$. In other words, the matrices M obey the condition that determines orthogonal groups. This group of $2n \times 2n$ matrices forms an *orthogonal* representation of the *unitary* group. It is a subgroup of $SO(2n)$. This matrix group is called $OU(2n)$. Symbolically,

$$U(n) \stackrel{C \rightarrow 2\times 2\ R}{\longrightarrow} OU(2n) \subset SO(2n) \tag{3.31}$$

There is an even more compelling reason to carry out the same type of replacement of quaternions by 2×2 complex matrices. Quaternions do not commute, as do real and complex numbers. Rather than worry about the order in which quaternions are written down in carrying out computations (such as constructing the determinant of a matrix), it is usually safer and more convenient to replace each quaternion in an $n \times n$ matrix by a 2×2 complex matrix using the embedding shown in Eq. (3.4). For the metric-preserving quaternion group $U(n;Q) = Sp(n)$ whose matrices obey $U^\dagger U = I_n$, this process generates $2n \times 2n$ complex matrices M that inherit the constraint in the form $M^\dagger M = I_{2n}$. In other words, the matrices M obey the condition that determines unitary groups (over C). This group of $2n \times 2n$ matrices forms a *unitary* representation of the *symplectic* group. It is a subgroup of $SU(2n)$. This matrix group is called $USp(2n)$. Symbolically,

$$Sp(n) \stackrel{Q \rightarrow 2\times 2\ C}{\longrightarrow} USp(2n) \subset SU(2n) \tag{3.32}$$

The groups $OU(2n)$ and $USp(2n)$ will appear in Chapter 11 (cf., Table 11.2) in the classification of the real forms of the simple Lie groups.

3.8 Modular Groups

We close with a useful aside. We have not considered matrices over the integers because they lack the geometric structure contributed by the continuous fields R, C, and Q. However, matrices over the integers play an important role in some areas of Lie group theory (representation theory of noncompact unimodular groups).

There are in fact three distinct groups over the integers that are sometimes confused:

(i) $GL(n;Z)$: If $m \in GL(n;Z)$, $\det(m) = \pm 1$.
(ii) $SL(n;Z)$: If $m \in SL(n;Z)$, $\det(m) = +1$.
(iii) $PSL(n;Z)$, n even: $PSL(n;Z) = SL(n;Z)/\{I_n, -I_n\}$.

For $n = 2$ these groups of matrices have the form $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$, with a, b, c, d all integers. If $\det(m) = n$, with n an integer, then $\det(m^{-1}) = 1/n$. Since the determinant of any matrix composed of integers must be an integer, the condition is that $\det(m) = \pm 1$. The subset of $GL(2;Z)$ with determinant $= +1$ forms the subgroup $SL(2;Z) \subset GL(2;Z)$. The modular group $PSL(2;Z)$ is obtained by identifying each pair of matrices in $SL(2;Z)$ of the form $\left[\begin{array}{cc} -a & -b \\ -c & -d \end{array}\right] \simeq \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$.

As a hint of the useful properties of these groups, we consider the matrix

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right] \in GL(n;Z) \tag{3.33}$$

Then

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right]^n = \left[\begin{array}{cc} F(n+1) & F(n) \\ F(n) & F(n-1) \end{array}\right] \tag{3.34}$$

where $F(n)$ is the nth Fibonacci number, defined recursively by

$$F(n) = F(n-1) + F(n-2)$$

n	0	1	2	3	4	5	6	7	$\cdots$
$F(n)$	0	1	1	2	3	5	8	13	$\cdots$

The proof by induction is simple. It proceeds by computation

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right]^{n+1} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{cc} F(n+1) & F(n) \\ F(n) & F(n-1) \end{array}\right] =$$

$$\begin{bmatrix} F(n+1)+F(n) & F(n)+F(n-1) \\ F(n+1) & F(n) \end{bmatrix} = \begin{bmatrix} F(n+2) & F(n+1) \\ F(n+1) & F(n) \end{bmatrix} \tag{3.35}$$

and by comparison of initial conditions for $n = 1$ [$F(0) = 0, F(1) = 1$]. Many other recursive relations among the integers are possible using different matrices in the groups $GL(2; Z)$, $GL(3; Z)$, etc.

The group $GL(n; Z)$ has important subgroups defined by imposing linear, quadratic, and multilinear constraints on the matrix elements, in exact analogy with $GL(n; R)$.

Imposing linear constraints generates subgroups with the structures given in Examples 2 through 7 above. The only remark necessary is that for the analogs of Example 5 (solvable groups) the diagonal matrix elements can only be ± 1.

Imposing quadratic constraints, for example $M^t I_n M = I_n$, generates a subgroup for which the sum of the squares of the matrix elements in each row or column is $+1$. Since the matrix elements themselves can only be $\pm 1, 0$, this group, $O(n; Z)$, consists of $n \times n$ matrices in which all but one matrix element in every row or column is zero, and the nonzero matrix element is ± 1. An important subgroup of $O(n; Z)$ is S_n, in which the nonzero matrix elements are all $+1$. This is the $n \times n$ faithful permutation representation P_n of the symmetric group S_n.

Finally, the multilinear condition $\det(m) = +1$ defines the unimodular subgroup $SL(n; Z)$ of $GL(n; Z)$.

Additional important groups are intersections of those just described. For example, the alternating group A_n consists of unimodular matrices in P_n:

$$A_n = P_n \cap SL(n; Z) \tag{3.36}$$

Example: The group $O(2; Z)$ consists of the $8 = 2^2 \times 2!$ matrices

$$\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.37}$$

The group $O(3, Z)$ has order $2^3 \times 3! = 48$. Its subgroup S_3 of order 6=3! consists of the six matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.38}$$

Its alternating subgroup $A_3 \subset S_3 \subset O(3, Z)$ consists of the three matrices with positive determinant, contained in the first row.

3.9 Conclusion

In this chapter we have taken advantage of a surprising observation: that most of the Lie groups encountered in applied (as well as pure) mathematics, the physical sciences, and the engineering disciplines are matrix groups. Most of the matrix groups typically encountered have been listed here. They consist of the general linear groups of $n \times n$ nonsingular matrices over the fields of real numbers, complex numbers, and quaternions, as well as subgroups obtained by imposing linear conditions, bilinear and quadratic conditions, and multilinear conditions on the matrix elements of the $n \times n$ matrices. Lie groups not encountered in the simple construction presented here consist primarily of some real forms (analytic continuations, encountered in Chapter 11) of those encountered here, the exceptional Lie groups G_2, F_4, E_6, E_7, E_8 and their real forms (encountered in Chapters 10 and 11), and covering groups of noncompact simple Lie groups such as $SL(2; R)$ (encountered in Chapter 7). We have in addition opened a door to analogs of Lie groups over the integers, $GL(n, Z)$, $SL(n; Z)$ and $PSL(n, Z)$. Matrix groups over finite fields are also of great interest, but fall outside the scope of our discussions.

3.10 Problems

1. Use the mapping (3.4) to construct a 2×2 matrix representation of the quaternions over the field of complex numbers. In particular, make the following associations, where $\mathcal{I}\mathcal{J} = -\mathcal{K}$:

$$\begin{array}{cccc} 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \mathcal{I} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mathcal{J} = i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & \mathcal{K} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \\ I_2 & i\sigma_x & i\sigma_y & i\sigma_z \end{array} \tag{3.39}$$

Here $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices, and i is the usual square root of -1. Show that any pair of the unit quaternions anticommute: i.e. $\{\mathcal{I}, \mathcal{J}\} = \mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I} = 0$.

2. Show that the unit quaternions $\mathcal{I}, \mathcal{J}, \mathcal{K}$ generate a group of order 8 under multiplication. Show that this group is isomorphic to $O(2; Z)$. Exhibit this isomorphism explicitly.

3. Show that $SU(1; Q) \sim SU(2; C)$.

4. Show that the dimensionalities (over the real field) of the general linear groups and their special linear subgroups are

$$\begin{array}{lclclcl} GL(n;R) & = & n^2 & \quad & SL(n;R) & = & n^2-1 \\ GL(n;C) & = & 2n^2 & \quad & SL(n;C) & = & 2n^2-2 \\ GL(n;Q) & = & 4n^2 & & & & \end{array}$$

5. Show that if the $n \times n$ metric matrix G is symmetric, nonsingular, and positive definite, then we can set $G = I_n$ in the definitions in Example 8. If the $n \times n$ metric matrix G is symmetric, nonsingular, and indefinite, then we can set $G = I_{p,q}$ in the definitions in Example 9, for suitable positive integers p and q, with $p + q = n$.

6. Show that it is possible to define subgroups $SL_i(n; C)$ of $GL(n; C)$ by the conditions

$$\begin{array}{llcll} SL_1(n;C): & \det(M) & = & e^{i\phi} & \quad 2n^2-1 \\ SL_2(n;C): & \det(M) & = & e^{\lambda} & \quad 2n^2-1 \\ SL_3(n;C): & \det(M) & = & r & \quad 2n^2-1 \\ SL(n;C): & \det(M) & = & +1 & \quad 2n^2-2 \end{array}$$

where ϕ, λ, r are real and $r \neq 0$. Show that the dimensions of these three subgroups are $2n^2 - 1$ and that $SL_3(n; C)$ is disconnected. It consists of two topologically identical copies of a $2n^2 - 1$ dimensional manifold, one of which contains the identity. Show that $SL(n; C) = SL_1(n; C) \cap SL_2(n; C)$. Do these results extend under: field restriction $C \rightarrow R$?; field extension $C \rightarrow Q$?

7. A subgroup of $UT(1, 1)$ includes matrices of the form $\begin{bmatrix} -1 & a \\ 0 & 1 \end{bmatrix}$, $a \in R$. Show that the underlying group manifold consists of two copies of the real line R^1. If matrices of the form $\begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ are also included, then the parameterizing manifold consists of how many copies of R^1?

8. Compute the dimensions of the real matrix groups in Examples (2)–(7) over the real field and show:

Group	Dimension
$UT(p,q)$	p^2+q^2+pq
$HT(p,q)$	$p(p+q)$
$UT(p,q,r)$	$p^2+q^2+r^2+pq+pr+qr$
$Sol(n)$	$n(n+1)/2$
$Nil(n)$	$n(n-1)/2$
$A(p,q)$	pq

What happens to these dimensions if the matrix groups are over the field of complex numbers? Quaternions?

9. Newton's equations of motion are $\mathbf{F} = d\mathbf{p}/dt$. In the Lorentz gauge Maxwell's equations can be written in the form

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A_\mu(x,t) = -\frac{4\pi}{c} j_\mu.$$

These equations can be expressed in a different coordinate system usisng either Galilean or Poincaré transformations. Verify that the equations do or do not remain invariant in form under these transformations, as follows:

Transformation	$\mathbf{F} = d\mathbf{p}/dt$	$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})A_\mu = -\frac{4\pi}{c}j_\mu$
Galilean	Invariant	Not Invariant
Poincare	Not Invariant	Invariant

How do you reconcile these results?

10. Show that the group of 2×2 matrices $SU(2)$ is parameterized by two complex numbers $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, so that $SU(2) = \begin{bmatrix} c_1 & c_2 \\ -c_2^* & c_1^* \end{bmatrix}$, subject to the condition $a_1^2 + b_1^2 + a_2^2 + b_2^2 = 1$. Convince yourself (a) that topologically this group (i.e., its parameterizing manifold) is equivalent to a three-sphere $S^3 \subset R^4$; and (b) algebraically it is equivalent to $SL(1;Q)$ [cf. (3.16)].

11. The group of 2×2 matrices $SU(1,1)$ is parameterized by two complex numbers $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, so that $SU(1,1) = \begin{bmatrix} c_1 & c_2 \\ c_2^* & c_1^* \end{bmatrix}$, subject to the condition $a_1^2 + b_1^2 - a_2^2 - b_2^2 = 1$. Identify the parameterizing manifold.

12. The group $SO(2)$ is one-dimensional. Show that every matrix in $SO(2)$ can be written in the form $\begin{bmatrix} m_{11} & x \\ m_{21} & m_{22} \end{bmatrix}$, where $m_{11}^2 + x^2 = 1$, so that $m_{11} = \pm\sqrt{1-x^2}$. The second row is orthogonal to the first, so that $m_{21}m_{11} + m_{22}x = 0$. As a result, we find

$$SO(2) \longrightarrow \begin{bmatrix} \pm\sqrt{1-x^2} & x \\ -x & \pm\sqrt{1-x^2} \end{bmatrix}$$

The $\pm$ signs are coherent. Each choice of sign ($\pm$) covers half the group.

13. The group $SO(3)$ is three-dimensional. Show that every matrix in $SO(3)$ can be written in the form

$$SO(3) \longrightarrow \begin{bmatrix} m_{11} & x & y \\ m_{21} & m_{22} & z \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Use arguments similar to those used in Problem 12 to express the matrix elements m_{ij} $i \geq j$ in terms of the three parameters (x, y, z).

14. An alternative parameterization of $SO(3)$ is given by

$$SO(3) \longrightarrow \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{Z_2}} & x \\ & & y \\ \hline -x & -y & Z_1 \end{array}\right] \times \left[\begin{array}{cc|c} \pm\sqrt{1-z^2} & z & 0 \\ -z & \pm\sqrt{1-z^2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Express the 2×2 and 1×1 submatrices Z_2 and Z_1 in terms of the coordinates (x, y). Determine the range of the parameters (x, y, z). How many square roots ("sheets") are necessary to completely cover $SO(3)$?

15. If $M \in GL(n; Z)$, show that $\det(M)$ must be ± 1.

16. Show that the orders of $O(n; Z) \supset S_n \supset A_n$ are $2^n \times n!,\ n!,\ \frac{1}{2}n!$.

17. Estimate the Fibonacci number $F(n)$ from the eigenvalues $\lambda_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$ of the generating matrix (3.33). What happens to this sequence if other initial conditions (than $F(0) = 0,\ F(1) = 1$) are introduced?

18. Derive other Fibonacci-type series using other symmetric generating matrices in $GL(2; Z)$ (for example, $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$) and other initial conditions.

19. The energy levels $|nlm\rangle$ of the nonrelativistic hydrogen atom exhibit an n^2-fold degeneracy under the Lie group $SO(4)$. All bound states with the same principle quantum number n have the same energy $E(nlm) = -E_0/n^2$ ($E_0 = 13.6$ eV). If the Coulomb symmetry is broken by placing one or more electrons in the Coulomb potential, the overall symmetry reduces to that of the rotation group: there is a symmetry reduction $SO(4) \downarrow SO(3)$. The representations of $SO(4)$ that enter into the description of the hydrogen atom bound states are indexed by the principle quantum number n ($n = 1, 2, 3 \ldots$). The $SO(4)$ representation with quantum number n splits into angular momentum representations that are indexed with quantum number l, $l = 0, 1, 2 \ldots, n-1$, with $\sum_{l=0}^{l=n-1}(2l+1) = n^2$. The $SO(3)$ multiplet with quantum number l is $2l+1$-fold degenerate. An empirical hamiltonian with $SO(4) \downarrow SO(3)$ broken symmetry that describes the filling order when electrons are introduced into a Coulomb potential established by a central charge $+Ze$ can be chosen to have the form:

$$E = -E_0 Z^2 \left\{1 + \delta * (n - l - 1)\right\} / n^2$$

This hamiltonian depends only on the quantum numbers of the representations of $SO(4)$ and its subgroup $SO(3)$. Show that this phenomenological energy spectrum with $\delta = 0.28$ provides the filling ordering that accounts for Mendelyeev's periodic table of the chemical elements: $(n, l) \rightarrow$ $1s;\ 2s, 2p;\ 3s, 3p;\ 4s, 3d, 4p; 5s, 4d, 5p;\ 6s, 4f, 5d, 6p;\ 7s, 5f, 6d, 7p;\ 8s, 6f, 7d, 8p; \ldots.$

20. **Symmetries:** Show the following equivalences:

$$\begin{array}{cc} UT(p,q) = UT(q,p) & SO(p,q) = SO(q,p) \\ A(p,q) = A(q,p) & U(p,q) = U(q,p) \\ & Sp(p,q) = Sp(q,p) \end{array}$$

21. G_1 and G_2 are two metrics on a real $2n$ dimensional linear vector space that are defined by

$$G_1 = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

Show that the $2n \times 2n$ matrices M that satisfy the bilinear constraints $M^t G_i M = G_i$ are:

G_1	G_2	G_1 and G_2
$O(2n;R)$	$Sp(2n;R)$	$OU(2n;R)$

22. In an n-dimensional linear vector space two coordinate systems x and y are related by a linear transformation: $y^j = x^i M_i{}^j$. Show that the derivatives are related by the same transformation (covariance - contravariance)

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} = M_i{}^j \frac{\partial}{\partial y^j}$$

As a result, a transformation that preserves a metric when acting on the coordinates preserves the same metric when acting on the derivatives.

23. The Poisson brackets between two functions $f(q,p)$ and $g(q,p)$ on a classical phase space of dimension $2n$ are defined by

$$\{f,g\} = \sum_k \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial p_k}$$

a. Show that these relations can be written in simple matrix form as

$$\{f,g\} = (Df)^t G(Dg) \quad \text{where} \quad G = \left[\begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array}\right] \quad \text{and} \quad (Dg) = \left[\begin{array}{c} \partial g/\partial q \\ \partial g/\partial p \end{array}\right]$$

b. Introduce a new coordinate system (Q,P), related to the original by a linear transformation of the form

$$\left[\begin{array}{c} \partial g/\partial Q \\ \partial g/\partial P \end{array}\right] = \left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\left[\begin{array}{c} \partial g/\partial q \\ \partial g/\partial p \end{array}\right]$$

Find the conditions on this $2n \times 2n$ matrix that preserves the structure of the Poisson brackets. Show A^tC and B^tD must be symmetric and $A^tD - B^tC = I_n$.

c. Show that the same conditions hold for linear transformations and the quantumn mechanical commutator bracket: $[q_j, q_k] = [p_j, p_k] = 0$ and $[q_j, p_k] = i\hbar\delta_{jk}$.

Note: The transformation from classical mechanics to quantum mechanics is made by identifying the classical Poisson bracket $\{,\}$ with the quantum commutator bracket $[,]$ according to

$$\{u(q,p), v(q,p)\} \leftrightarrow \frac{[u(\hat{q},\hat{p}), v(\hat{q},\hat{p})]}{i\hbar}$$

The hat ˆ indicates an operator.

24. **Transfer Matrices:** Figure 3.2 shows a potential in one dimension. The wave function to the left of the interaction region has the form

$$\psi_L(x) = A_L e^{+ikx} + B_L e^{-ikx} = \left[\begin{array}{cc} e^{+ikx} & e^{-ikx} \end{array} \right] \left[\begin{array}{c} A_L \\ B_L \end{array} \right]$$

with a similar expression for the wavefunction on the right. The exponential e^{+ikx} describes a particle of mass m moving to the right $(+)$ with momentum $\hbar k$ and energy $E = (\hbar k)^2/2m$. The complex number A_L is the probability amplitude for finding a particle moving to the right with this momentum. The expected value of the momentum in the left-hand region is $\langle \hat{p} \rangle = (|A_L|^2 - |B_L|^2)\hbar k$, where the operator $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$.

a. Show that conservation of momentum leads to the equation

$$|A_L|^2 - |B_L|^2 = |A_R|^2 - |B_R|^2$$

when the asymptotic value of the potential to the left of the interaction region is the same as the value on the right.

b. Since the Schrödinger equation is second order the four amplitudes $A_L, A_R,\ B_L, B_R$ are not independent. Only two are independent. Two linear relations exist among them. Show that they can be expressed in terms of a matrix relation of the form

$$\left[\begin{array}{c} A_L \\ B_L \end{array} \right] = \left[\begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array} \right] \left[\begin{array}{c} A_R \\ B_R \end{array} \right]$$

The 2×2 matrix T is called a *transfer matrix*. The transfer matrix is a function of energy E. Show that $T(E) \in U(1,1)$.

c. Show that $T \in SU(1,1)$ by appropriate choice of phase.

25. **Crossing Symmetry:** A transfer matrix T for a one dimensional potential relates amplitudes for the wavefunction on the left of the interaction region with the amplitudes on the right. A scattering matrix (S-matrix) S relates the incoming amplitudes with the outgoing amplitudes:

$$\left[\begin{array}{c} A_L \\ B_L \end{array} \right] = T \left[\begin{array}{c} A_R \\ B_R \end{array} \right] \qquad\qquad \left[\begin{array}{c} A_R \\ B_L \end{array} \right] = S \left[\begin{array}{c} A_L \\ B_R \end{array} \right]$$

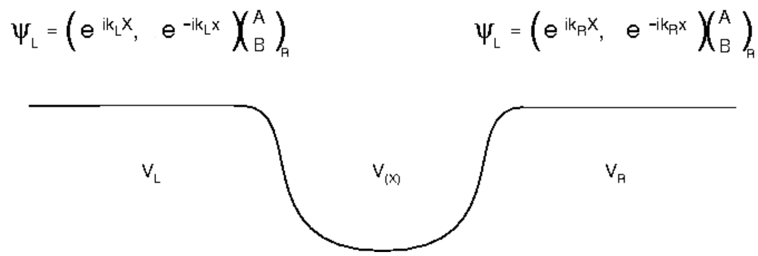

Fig. 3.2. The potentials to the left and right of the interaction region are constant, with $V_L = V_R$. The wave functions to the left and right of this region are represented in the form $\psi_\sigma(x) = A_\sigma e^{+ikx} + B_\sigma e^{-ikx}$, where $\sigma = L, R$.

a. Invoke conservation of momentum arguments to conclude $S \in U(2)$.

b. Show that the matrix elements of S and T are related by

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{t_{11}} & -\dfrac{t_{12}}{t_{11}} \\[2ex] \dfrac{t_{21}}{t_{11}} & \dfrac{t_{11}t_{22} - t_{12}t_{21}}{t_{11}} \end{bmatrix}$$

c. Show that the poles of $S(E)$ are the zeroes of $T(E)$, specifically of $t_{11}(E)$. Poles along the real energy axis describe bound states. Poles off the real axis of the form $r_j/[(E - E_j) + i(\Gamma_j/2)]$ describe resonances at energy E_j with characteristic decay time $\Gamma_j/\hbar$.

26. Two interaction regions V_1 and V_2 on the line are characterized by transfer matrices T_1 and T_2, and also by S-matrices S_1 and S_2 (c.f., Fig. 3.3). The outputs of one region are inputs to the other, as follows:

$$\begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} o_1 \\ o_4 \end{bmatrix}$$

a. The transfer matrices for the two regions are defined by

$$\begin{bmatrix} i_1 \\ o_2 \end{bmatrix} = T_1 \begin{bmatrix} o_1 \\ i_2 \end{bmatrix} \qquad \begin{bmatrix} i_3 \\ o_4 \end{bmatrix} = T_2 \begin{bmatrix} o_3 \\ i_4 \end{bmatrix}$$

Show that the transfer matrix for the entire interaction region is

$$\begin{bmatrix} i_1 \\ o_2 \end{bmatrix} = T_{\mathrm{Tot}} \begin{bmatrix} o_3 \\ i_4 \end{bmatrix} \qquad T_{\mathrm{Tot}} = T_1 T_2$$

b. The S-matrices for the two regions relate inputs to outputs as follows

$$\begin{bmatrix} o_1 \\ o_2 \\ o_3 \\ o_4 \end{bmatrix} \left[\begin{array}{cc|cc} s_{11} & s_{12} & 0 & 0 \\ s_{21} & s_{22} & 0 & 0 \\ \hline 0 & 0 & s_{33} & s_{34} \\ 0 & 0 & s_{43} & s_{44} \end{array}\right] \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

Show that the scattering matrix for the entire region is

$$S_{\mathrm{Tot}} = \begin{bmatrix} 0 & s_{34} \\ s_{21} & 0 \end{bmatrix} + \frac{1}{1 - s_{12}s_{43}} \begin{bmatrix} s_{33}s_{22} & s_{33}s_{112}s_{44} \\ s_{22}s_{43}s_{11} & s_{22}s_{44} \end{bmatrix}$$

c. Show that S_{Tot} is unitary.

d. Interpret the matrix S_{Tot} in terms of Feynman-like sum over all paths. Do this by expanding the fraction $1/(1 - s_{12}s_{43})$ as a geomeetric sum and interpreting each term in this expansion as a path through the two scattering potentials.

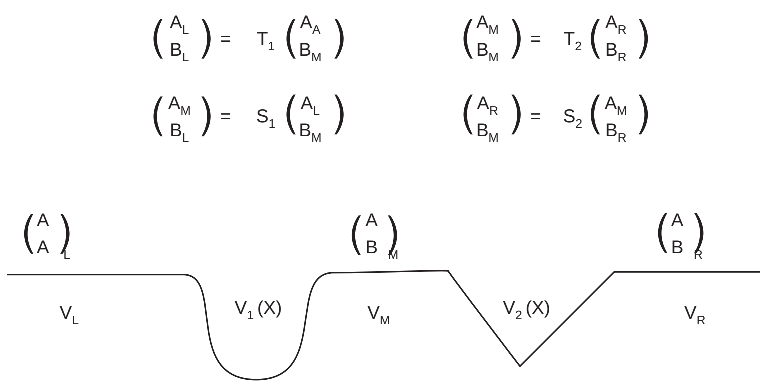

Fig. 3.3. Two potentials on the line are characterized by their T and S matrices.

27. If the potentials V_1 and V_2 are modified to $V_1{}'$ and $V_2{}'$ their transfer matrices and their scattering matrices will also be modified $T_i(E) \to T_i^{'}(E)$ and $S_i(E) \to S_i^{'}(E)$, $i = 1, 2$. It is possible that for

some energy E, $S'_{\text{Tot}}(E) = S_{\text{Tot}}(E)$. Find the set of all modified scattering matrices $S'_1(E)$ and $S'_2(E)$ with the property that the modified pair maps into the original total S-matrix $S_{\text{Tot}}(E)$. In fancy terms, find the fiber in $U(4) \supset U(2) \otimes U(2) \downarrow U(2)$. **Hint:** If this seems daunting, note that to satisfy $T_1(E)T_2(E) = T_{\text{Tot}}(E) = T'_1(E)T'_2(E)$ we can take $T'_1(E) = T_1(E)R$ and $T'_2(E) = R^{-1}T_2(E)$ for any $R \in U(1,1)$. The fiber in $U(2,2) \supset U(1,1) \otimes U(1,1) \downarrow U(1,1)$ over $T_{\text{Tot}}(E)$ consists of the matrices $(T_1(E)R, R^{-1}T_2(E))$. Now map this into the fiber $(S'_1(E), S'_2(E))$ over $S_{\text{Tot}}(E)$.

28. A passive linear device, classical or quantum, can be described by an S matrix. If the device has n external leads the scattering matrix is an $n \times n$ matrix. Devices with $n_1, n_2, \cdots, n_k$ leads can be connected together by soldering some of the leads together. The leads that are soldered together are the *internal* leads. The remainder of the leads are *external* leads. We distinguish between internal and external leads by subscripts i and e. The S matrix that describes the original set of k devices is a direct sum of k S matrices of sizes $n_j \times n_j$ $(j = 1, 2, ..., k)$. Through appropriate permutation of the rows and columns of this direct sum of S matrices the input-output relations can be expressed in the form

$$\begin{bmatrix} i_i \\ i_e \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} o_i \\ o_e \end{bmatrix} \qquad\qquad [o_i] = \Gamma\,[i_i]$$

The matrix Γ that relates internal outputs to internal inputs describes the topology, or connectivity, of the network.

a. Show that the S matrix that describes the network, defined by $[o_e] = S_{\text{Network}}\,[i_e]$, is given by (c.f., Problem 3.26**c.**)

$$S_{\text{Network}} = D + C\Gamma(I - A\Gamma)^{-1}B$$

b. Show that S_{Newtork} is unitary: $S^{\dagger}_{\text{Newtork}} = S_{\text{Newtork}}$, $S_{\text{Newtork}} \subset U(d)$.

c. Expand S_{Newtork} to show that

$$S_{\text{Network}} = D + C\Gamma B + C\Gamma A\Gamma B + C\Gamma A\Gamma A\Gamma B + C\Gamma A\Gamma A\Gamma A\Gamma B + ...$$

Interpret this expansion in terms of a Feynman sum over all possible scattering paths through the network.

30. A mathematical description of the preceeding problem involves a subgroup restriction $U(\sum_{j=1}^{k} n_j) \supset \Pi_{j=1}^{k} \otimes U(n_j)$ and a projection

to the total network scattering matrix in $U(d)$, where d is the number of the network's external leads. The connectivity is determined by the permutation matrix Γ. Determine the fiber in $\Pi_{j=1}^{k} \otimes U(n_j)$ over each group operation in $U(d)$.

31. All the matrices in this problem are square $n \times n$, with: H hermitian; U unitary; A antihermitian. Show the right hand columns follow from the definition in the left hand column.

$$
\begin{array}{ccc}
H_2 = \dfrac{H_1 + I_n}{H_1 - I_n} & [H_1, H_2] = 0 & H_1 = \dfrac{H_2 + I_n}{H_2 - I_n} \\[2ex]
U = \dfrac{I_n + iH}{I_n - iH} & [H, U] = 0 & H = i\dfrac{I_n - U}{I_n + U} \\[2ex]
A = \dfrac{I_n + iU}{I_n - iU} & [U, A] = 0 & U = i\dfrac{I_n - A}{I_n + A} \\[2ex]
H = \dfrac{I_n - iA}{I_n + iA} & [A, H] = 0 & A = i\dfrac{H - I_n}{H + I_n}
\end{array}
$$