Classical Special Functions and Lie Groups

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Abstract

The classical orthogonal functions of mathematical physics are closely related to Lie groups. Specifically, they are matrix elements of, or basis vectors for, unitary irreducible representations of low-dimensional Lie groups. We illustrate this connection for: The Wigner functions, spherical harmonics, and Legendre polynomials; the Bessel functions; and the Hermite polynomials. These functions are associated with the Lie groups: the rotation group $SO(3)$ in three-space and its covering group $SU(2)$; the Euclidean group in the plane $E(2)$ or $ISO(2)$; and the Heisenberg group H_4.

1 Introduction

There are many different approaches to understanding the classical special functions of Mathematical Physics. These include: solutions to second order ordinary differential equations; orthogonal polynomials (excluding Bessel and analogous functions); expressions obtained from contour integrals in the complex plane; integral transforms of other functions; specializations of generalized hypergeometric or related series; connections with Lie groups. All but the last point of view are well-served by texts or other sources that are accessible at the undergraduate level [1, 2, 3, 4, 5, 6, 7, 8, 9]. Excellent expositions of the relation between Lie group theory and the special functions exist at the advanced level since 1968 [10, 11, 12, 13] but even today there is no source for this information that is accesssible at the undergraduate level.

The purpose of the present work is to make this connection clear at a more elementary level. We do this by briefly providing an overview of the details of this connection, and then providing three explicit examples. The examples begin with a choice of group — in the present work the groups are the rotation group, the Euclidean group in the plane, and the Heisenberg group. In each case we introduce the group as a simple, low-dimensional matrix group depending on a suitable number of parameters. Then spaces are introduced on which the group acts. These are spaces of functions depending on continuous variables, in which case the group acts through differential operators. They also include spaces with a discrete set of basis vectors, in which case the group acts through its matrix representations. The connection between these spaces is equivalent to an overlap integral — typically depending on some continuous variables and an equal number of discrete variables. These discretely indexed functions are the classical special functions of mathematical physics. All their properties are reflections of the properties of the representations of these groups and of the group operations themselves.

An overview is presented in Sec. 2. Applications to the rotation group $SO(3)$ and its covering group $SU(2)$ are presented in Sec. 3; to the Euclidean group in the plane $E(2)$ or $ISO(2)$ are presented in Sec. 4; and to the Heisenberg group H_3 or H_4 are presented in Sec. 5. In the closing Section (Sec. 6) we provide a brief summary of some of the steps taken since the creation of Wave Mechanics in 1926 provoked the evolution of our understanding of this aspect of the theory of special functions. This work grew out of an undergraduate thesis by one of the authors [14].

2 Overview

The operations in a Lie group G can be parameterized by points $\mathbf{a} \in R^n$ in an n-dimensional manifold ($n = 3$ or 4 in this work): $g(\mathbf{a}) \in G$. It is usual to parameterize the identity I by the point with coordinates $\mathbf{0}$, so that group operations near the identity can be represented in the form $g(\delta\mathbf{a}) = I + \delta a_i X_i + \text{h.o.t.}$ The operators X_i $(i = 1, 2, \cdots, n)$ are obtained by *linearization* of the group operation in the neighborhood of the identity. The *infinitesimal generators* X_i close under commutation, $[X_i, X_j] = c_{ij}^{\ \ k} X_k$, and the constants $c_{ij}^{\ \ k}$ are called structure constants. The operators X_i span the Lie algebra $\mathfrak{g}$ of the Lie group G.

Operations $g(\mathbf{a}) \in G$ can be recovered by inverting the linearization process. This is done via the exponential mapping:

$$g(\mathbf{a}) = e^{\mathbf{a}\cdot\mathbf{X}} = e^{\sum a_i X_i} \tag{1}$$

In this parameterization $g^{-1}(\mathbf{a}) = g(-\mathbf{a})$.

The group operations $g(\mathbf{a})$ can be mapped isomorphically $(1 : 1$, invertible) or homomorphically $(2 : 1$ or $3 : 1$ etc., locally invertible) to matrices acting on a linear vector space with a discrete basis $|\mathbf{n}\rangle$:

$$g(\mathbf{a})|\mathbf{n}\rangle = |\mathbf{n}'\rangle\langle\mathbf{n}'|g(\mathbf{a})|\mathbf{n}\rangle \tag{2}$$

These matrices provide a representation of the group. If the mapping is 1:1 the representation is faithful. If the set of matrices representing a group cannot be simultaneously block diagonalized, the matrix representation is irreducible. This matrix representation is unitary if

$$\langle\mathbf{n}'|g^{-1}(\mathbf{a})|\mathbf{n}\rangle = \langle\mathbf{n}|g(\mathbf{a})|\mathbf{n}'\rangle^* \tag{3}$$

The matrix elements are functions of discrete indices $\mathbf{N} = (\mathbf{n}', \mathbf{n})$ as well as the continuous variables $\mathbf{a}$. It is useful to emphasize the duality between the discrete and continuous varibles by writing these functions in the Dirac notation

$$\langle\mathbf{n}'|g(\mathbf{a})|\mathbf{n}\rangle = \langle\mathbf{a}|\mathbf{N}\rangle = f_{\mathbf{N}}(\mathbf{a}) \tag{4}$$

Here $f_{\mathbf{N}}(\mathbf{a})$ is one of the special functions.

The usual properties of the classical special functions can all be expressed by computing suitable operators in a mixed basis in two ways:

$$\begin{array}{ccc} & \langle\mathbf{a}|\mathcal{O}|\mathbf{N}\rangle & \\ \swarrow & & \searrow \\ \langle\mathbf{a}|\mathcal{O}|\mathbf{a}'\rangle\langle\mathbf{a}'|\mathbf{N}\rangle & = & \langle\mathbf{a}|\mathbf{N}'\rangle\langle\mathbf{N}'|\mathcal{O}|\mathbf{N}\rangle \end{array} \tag{5}$$

The summation convention (sums / integrals occur over repeated indices) is adopted throughout.

On the left the operator $\langle\mathbf{a}|\mathcal{O}|\mathbf{a}'\rangle$ is expressed in the continuous basis. It is typically a first- or second- order differential operator, acting on the special function $f_{\mathbf{N}}(\mathbf{a}')$. On the right the operator $\langle\mathbf{N}'|\mathcal{O}|\mathbf{N}\rangle$ is a matrix element, depending on the discrete indices $\mathbf{N}', \mathbf{N}$, and $\langle\mathbf{a}|\mathbf{N}'\rangle\langle\mathbf{N}'|\mathcal{O}|\mathbf{N}\rangle$ is a sum over special functions evaluated at point $\mathbf{a}$.

In the cases of interest, it is always possible to choose operators X_i, $i = 1, 2, 3, \cdots$ so that $[X_3, X_1 \pm iX_2] = \pm(X_1 \pm iX_2)$. The three different Lie groups that we study differ in the value of the commutator $[X_1, X_2]$.

The properties of the special functions are related to the choice of operator $\mathcal{O}$ as follows:

Table 1: Some Lie groups and the special functions associated with them.

Lie Group(s)	**Special Function(s)**
Special Unitary Group $SU(2)$	Wigner Functions, Jacobi Polynomials
Rotation Group $SO(3)$	Spherical Harmonics, Legendre Polynomials
Euclidean Group $ISO(2), E(2)$	Bessel Functions
Heisenberg Groups H_3, H_4	Hermite Polynomials
Special Linear Group $SL(2;R)$	Hypergeometric Functions
Special Orthogonal Group $SO(n)$	Gegenbauer Functions
Third-order Triangular Matrices	Laguerre Polynomials, Whittaker Functions

Second order differential equation: $\mathcal{O}$ is an operator quadratic in the infinitesimal generators X_i ("Casimir operator") that commutes with all infinitesimal generators: $[\mathcal{O}, X_i] = 0$.

Generating Functions: $\mathcal{O} = e^{tX_\pm}$.

Recurrence Relations: $\mathcal{O} = X_+ + X_-$.

Differential Relations: $\mathcal{O} = X_+ - X_-$

Addition theorems are a direct consequence of the group multiplication law:

$$\begin{array}{ccc} & \langle \mathbf{n}'|g(\mathbf{a})g(\mathbf{b})|\mathbf{n}\rangle & \\ \swarrow & & \searrow \\ \langle \mathbf{n}'|g(\mathbf{a})|\mathbf{n}''\rangle\langle \mathbf{n}''|g(\mathbf{b})|\mathbf{n}\rangle & = & \langle \mathbf{n}'|g(\mathbf{c})|\mathbf{n}\rangle \end{array} \tag{6}$$

where $\mathbf{c}$ is some function of the coordinates $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{c} = h(\mathbf{a}, \mathbf{b})$.

A more complete picture of the relation between some Lie groups and some special functions is presented in Table 1.

We now illustrate some details of these relations for the three groups: $SU(2) \cong SO(3)$; $E(2) = ISO(2)$; H_4.

3 SU(2) and SO(3)

The rotation group $SO(3)$ with infinitesimal generators L_j and the group $SU(2)$ with infinitesimal generators $S_j = \frac{i}{2}\sigma_j$ (σ are the Pauli spin matrices) have isomorphic commutation relations. Therefore the two groups are closely related: $SO(3)$ is a $2:1$ image of $SU(2)$ [15]. As a result, all the unitary irreducible representations of $SO(3)$ are obtained as a subset of all the unitary irreducible representations of $SU(2)$. These in turn are obtained as symmetrized tensor products of the smallest faithful unitary representation, consisting of 2×2 matrices. These matrices, which are called Wigner rotation matrices, can be parameterized by three rotation angles (ϕ,θ,ψ). The Wigner rotation matrices $D^j_{m'm}(\phi,\theta,\psi)$ have been used in physics for a long time [15]. We begin our discussion of the connection between Lie groups and special functions with a description of the connection of the matrix elements of the Wigner rotation matrices with the Wigner functions, the spherical harmonics, and the Legendre polynomials.

3.1 Rotation Group Matrix Elements

The rotation group has a simple parameterization in terms of Euler angles when acting in R^3:

$$R(\phi,\theta,\psi) = e^{-i\phi J_z} e^{-i\theta J_y} e^{-i\psi J_z} \tag{7}$$

where J_x, J_y, J_z describe rotations about the x-, y-, and z-axes, respectively. Their commutation relations, when expressed in terms of the angular momentum operators $L_i = \epsilon_{ijk} x_j (\hbar/i)\partial_k$ (where $i = 1,2,3$ or x,y,z and $\hbar = 1$) are

$$\begin{array}{rcl} [J_x, J_y] & = & iJ_z \\ [J_y, J_z] & = & iJ_x \\ [J_z, J_x] & = & iJ_y \end{array} \tag{8}$$

The second order operator $J^2 = J_x^2 + J_y^2 + J_z^2$ commutes with all J_i, and so has constant value in any irreducible representation of this group.

Irreducible representations are constructed by introducing basis vectors $\left| \begin{array}{c} j \\ m \end{array} \right\rangle$, with $2j = 0,1,2,\cdots$ and $m = -j, -j+1, \cdots, +j$.

In a $2j+1$ dimensional irreducible representation the matrix elements of $J^2, J_z, J_\pm = J_x \pm iJ_y$ are

$$\begin{aligned}
\langle \begin{smallmatrix} j' \\ m' \end{smallmatrix} |J^2| \begin{smallmatrix} j \\ m \end{smallmatrix} \rangle &= j(j+1)\delta^{j'j}\delta_{m'm} \\
\langle \begin{smallmatrix} j' \\ m' \end{smallmatrix} |J_z| \begin{smallmatrix} j \\ m \end{smallmatrix} \rangle &= m\delta^{j'j}\delta_{m'm} \\
\langle \begin{smallmatrix} j' \\ m' \end{smallmatrix} |J_+| \begin{smallmatrix} j \\ m \end{smallmatrix} \rangle &= \sqrt{(j-m)(j+m')}\delta^{j'j}\delta_{m',m+1} \\
\langle \begin{smallmatrix} j' \\ m' \end{smallmatrix} |J_-| \begin{smallmatrix} j \\ m \end{smallmatrix} \rangle &= \sqrt{(j+m)(j-m')}\delta^{j'j}\delta_{m',m-1}
\end{aligned} \tag{9}$$

The action of the rotation operation Eq. (7) on the basis vectors $| \begin{smallmatrix} j \\ m \end{smallmatrix} \rangle$ is given by

$$R(\phi,\theta,\psi)| \begin{smallmatrix} j \\ m \end{smallmatrix} \rangle = | \begin{smallmatrix} j \\ m' \end{smallmatrix} \rangle\langle \begin{smallmatrix} j \\ m' \end{smallmatrix} |R(\phi,\theta,\psi)| \begin{smallmatrix} j \\ m \end{smallmatrix} \rangle = | \begin{smallmatrix} j \\ m' \end{smallmatrix} \rangle D^j_{m'm}(\phi,\theta,\psi) \tag{10}$$

The matrix elements of the Wigner D matrix are

$$D^j_{m'm}(\phi,\theta,\psi) = e^{-im'\phi} d^j_{m'm}(\theta) e^{-im\psi} \tag{11}$$

and the matrix elements of the Wigner (little) d matrix are

$$d^j_{m'm}(\theta) = \left(e^{-i\theta J_y}\right)_{m'm} = \left[(j+m')!(j-m')!(j+m)!(j-m)!\right]^{1/2}$$

$$\sum_s \frac{(-1)^{m'-m+s}}{s!(m'-m-s)!(j+m-s)!(j-m'-s)!} \left(\cos(\frac{\theta}{2})\right)^{2j+m-m'-2s} \left(\sin(\frac{\theta}{2})\right)^{m'-m+2s} \tag{12}$$

The inverse of a group operation is

$$\left[R(\phi,\theta,\psi)\right]^{-1} = R(-\psi,-\theta,-\phi) \tag{13}$$

The matrix inverse of a unitary representation Eq.(10) is its hermitian adjoint, or complex conjugate transpose

$$\left[D^j(-\psi,-\theta,-\phi)\right]_{nm} = \left[D^j(\phi,\theta,\psi)\right]^*_{mn} \tag{14}$$

The simplest faithful matrix representation, with $j = \frac{1}{2}$, is

$$\left[\begin{array}{cc} e^{-i\phi/2}\cos\theta/2e^{-i\psi/2} & -e^{-i\phi/2}\sin\theta/2e^{+i\psi/2} \\ e^{+i\phi/2}\sin\theta/2e^{-i\psi/2} & e^{+i\phi/2}\cos\theta/2e^{+i\psi/2} \end{array}\right] \tag{15}$$

3.2 Orthogonality and Completeness

The Wigner matrix elements Eq. (10) form an orthogonal and complete set of functions defined on the continuous space that parameterizes this group. The parameters are $(0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi, 0 \leq \psi \leq 2\pi)$ and the measure (volume element) on this space is $dV = d\phi|d\cos(\theta)|d\psi = \sin(\theta)d\theta d\phi d\psi$. Under the identification motivated by the clean lines of the Dirac notation, the Wigner functions are defined as follows

$$\langle \phi, \theta, \psi| \begin{matrix} j \\ m'm \end{matrix} \rangle = \sqrt{\frac{2j+1}{8\pi^2}} D^j_{m'm}(\phi, \theta, \psi) = \langle \begin{matrix} j \\ m'm \end{matrix} |\phi, \theta, \psi\rangle^* \tag{16}$$

The orthogonality and completeness relations can be expressed

$$\begin{array}{rcll} \langle \begin{matrix} j' \\ n'm' \end{matrix} |\phi, \theta, \psi\rangle\langle\phi, \theta, \psi| \begin{matrix} j \\ nm \end{matrix} \rangle & = & \langle \begin{matrix} j' \\ n'm' \end{matrix} | \begin{matrix} j \\ nm \end{matrix} \rangle & \text{Orthogonality} \\ \langle\phi', \theta', \psi'| \begin{matrix} j \\ m'm \end{matrix} \rangle\langle \begin{matrix} j \\ m'm \end{matrix} |\phi, \theta, \psi\rangle & = & \langle\phi', \theta', \psi'|\phi, \theta, \psi\rangle & \text{Completeness} \end{array} \tag{17}$$

On the left hand side of these equations an integral over the continuous variables (ϕ, θ, ψ) is assumed using the measure dV in the first equation and a sum over all allowed discrete indices (j, m', m) is assumed in the second line. On the right hand side the inner products of the discrete and continuous bras and kets are

$$\begin{array}{rcl} \langle \begin{matrix} j' \\ n'm' \end{matrix} | \begin{matrix} j \\ nm \end{matrix} \rangle & = & \delta^{j'j}\delta_{n'm'}\delta_{nm} \\ \langle\phi', \theta', \psi'|\phi, \theta, \psi\rangle & = & \delta(\phi' - \phi)\delta(\cos\theta' - \cos\theta)\delta(\psi' - \psi) \end{array} \tag{18}$$

3.3 Spherical Harmonics and Legendre Polynomials

The subset of Wigner functions Eq. (16) that do not depend on the rotation angle ψ have $m = 0$, and therefore $j = l$, an integer. These functions are proportional to the spherical harmonics, and in fact

$$\langle \theta, \phi | \begin{array}{c} l \\ m' \end{array} \rangle = Y^l_{m'}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} D^l_{m',0}(\phi, \theta, -)^* \tag{19}$$

These functions inherit their orthonormality and completeness properties on the surface of the two-dimensional sphere (θ, ϕ) from the corresponding properties of their parents, the Wigner functions, on the original three-dimensional parameter space.

The subset of these functions that do not depend on the rotation angle ϕ have $m' = 0$, and are Legendre polynomials:

$$D^l_{00}(-, \theta, -) = P_l(\cos\theta) \tag{20}$$

They are orthogonal and complete on the meridian $(0 \leq \theta \leq \pi)$ Their normalization is obtained from the Wigner functions:

$$\int_0^\pi P_{l'}(\cos\theta) P_l(\cos\theta) \sin\theta d\theta = \frac{2}{2l+1} \delta_{l'l} \tag{21}$$

3.4 Addition Theorems

The product of two group operations is a group operation: $R(\mathbf{a})R(\mathbf{b}) = R(\mathbf{c})$ where, for example, $\mathbf{a} = (\phi_1, \theta_1, \psi_1)$, $R(\mathbf{a})$ is given in Eq. (7), and similarly for $\mathbf{b}$, and the three variables of $\mathbf{c} = \mathbf{c}(\mathbf{a}, \mathbf{b})$ are functions of the parameters in the triplets $\mathbf{a}$ and $\mathbf{b}$. The explicit expressions for (Φ, Θ, Ψ) (e.g., $\Theta(\phi_1\theta_1\psi_1; \phi_2\theta_2\psi_2)$), can be computed in any faithful matrix representation, for example the 2×2 matrix representation given in Eq. (15).

In any matrix representation

$$\begin{array}{ccc}
 & D^j_{m'm}\left(R(\mathbf{a})R(\mathbf{b})\right) & \\
\swarrow & & \searrow \\
D^j_{m'k}\left(R(\mathbf{a})\right) D^j_{km}\left(R(\mathbf{b})\right) & = & D^j_{m'm}\left(R(\mathbf{c})\right) \\
\| & & \| \\
\langle \mathbf{a} | \begin{array}{c} j \\ m'k \end{array} \rangle \langle \mathbf{b} | \begin{array}{c} j \\ km \end{array} \rangle & = & \sqrt{\frac{2j+1}{8\pi^2}} \langle \mathbf{c} | \begin{array}{c} j \\ m'm \end{array} \rangle
\end{array} \tag{22}$$

When the specific angles described by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are introduced in this expression, the resulting addition theorem in the group manifold is obtained for the Wigner matrix elements.

Most of these addition formulas are of little use. However, by setting $m' = m = 0$ we find

$$D^l_{0k}(-, -\theta_1, -\phi_1) D^l_{k0}(\phi_2, \theta_2, -) = D^l_{00}(-, \Theta, -) \tag{23}$$

where

$$\cos\Theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1) \tag{24}$$

By using the relation between the matrix elements D^j_{m0} and the spherical harmonics (c.f., Eq. (19)) and the Legendre polynomials (c.f., Eq. (20)) we find the sometimes-useful result

$$P_l(\cos\Theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y^l_m(\theta_1, \phi_1) Y^{l*}_m(\theta_2, \phi_2) \tag{25}$$

3.5 Generating Functions

There are many generating functions. We can compute one by investigating the operator describing rotations about the y-axis in a mixed basis

$$\begin{array}{ccc} & \langle\theta,\phi|e^{-i\beta J_y}|\begin{matrix} l \\ -l \end{matrix}\rangle & \\ \swarrow & & \searrow \\ \langle\theta,\phi|e^{-i\beta J_y}|\theta',\phi'\rangle\langle\theta',\phi'|\begin{matrix} l \\ -l \end{matrix}\rangle & = & \langle\theta,\phi|\begin{matrix} l \\ m' \end{matrix}\rangle\langle\begin{matrix} l \\ m' \end{matrix}|e^{-i\beta J_y}|\begin{matrix} l \\ -l \end{matrix}\rangle \end{array} \tag{26}$$

The transformation on the left can be computed using the 3×3 matrix representation for the rotation and the coordinates $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$ so that

$$\left[\begin{array}{ccc} x & y & z \end{array}\right] = \left[\begin{array}{ccc} x' & y' & z' \end{array}\right] \left[\begin{array}{ccc} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{array}\right] \tag{27}$$

Solving for θ', ϕ' in terms of θ, ϕ, we find

$$\sin\theta' e^{-i\phi'} = \sin\theta e^{-i\phi} - \sin\beta\cos\theta + (\cos\beta - 1)\sin\theta\cos\phi \tag{28}$$

Using the definition of $Y^l_{-l}(\theta,\phi) = \sqrt{\frac{(2l+1)!!}{4\pi(2l)!!}}\left(\sin\theta e^{-i\phi}\right)^l$, where $n!! = n(n-2)(n-4)\cdots 2$ or 1, we find on the left hand side of Eq. (26)

$$\mathrm{LHS}(26) = \sqrt{\frac{(2l+1)!!}{4\pi(2l)!!}}\left(\sin\theta e^{-i\phi} - \sin\beta\cos\theta + (\cos\beta - 1)\sin\theta\cos\phi\right)^l \tag{29}$$

The right hand side of Eq. (26) can be treated by constructing a disentangling relation:

$$e^{-i\beta J_y} = e^{\alpha_+ L_+} e^{\alpha_z L_z} e^{\alpha_- L_-} \tag{30}$$

Analytyic expressions for the three variables α_* are determined by carrying out this calculation in any faithful matrix representation. The 2×2 matrix representation of Eq. (15) is the smallest such representation, so the calculations in this representation are the simplest. By setting

$$\begin{bmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{bmatrix} = \begin{bmatrix} 1 & \alpha_+ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\alpha_z/2} & 0 \\ 0 & e^{-\alpha_z/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha_- & 1 \end{bmatrix} \tag{31}$$

we find

$$\alpha_- = \tan\beta/2 \qquad \alpha_+ = -\tan\beta/2 \qquad e^{-\alpha_z/2} = \cos\beta/2 \tag{32}$$

When this operator is applied to the state $\left|\begin{array}{c} l \\ -l \end{array}\right\rangle$ we find

$$\begin{aligned} e^{-i\beta J_y}\left|\begin{array}{c} l \\ -l \end{array}\right\rangle &= e^{\alpha_+ L_+} e^{\alpha_z L_z} e^{\alpha_- L_-}\left|\begin{array}{c} l \\ -l \end{array}\right\rangle \\ &= e^{\alpha_+ L_+} e^{\alpha_z L_z}\left|\begin{array}{c} l \\ -l \end{array}\right\rangle \\ &= e^{\alpha_+ L_+}\left|\begin{array}{c} l \\ -l \end{array}\right\rangle e^{(-l)\alpha_z} \\ &= \sum_k \frac{(\alpha_+ L_+)^k}{k!}\left|\begin{array}{c} l \\ -l \end{array}\right\rangle e^{2l(-\alpha_z/2)} \\ &= \sqrt{\frac{(2l)!}{(2l-k)!k!}}\left|\begin{array}{c} l \\ -l+k \end{array}\right\rangle (-\tan\beta/2)^k (\cos\beta/2)^{2l} \end{aligned} \tag{33}$$

As a result, the right hand side of Eq. (26) is

$$\text{RHS}(26) = \sqrt{\frac{(2l)!}{(l+m)!(l-m)!}}(-\sin\beta/2)^{l+m}(\cos\beta/2)^{l-m}Y^l_m(\theta,\phi) \tag{34}$$

The desired generating function is obtained by equating Eq. (29) with Eq. (34).

3.6 Differential Operators

Differential operators can be introduced that act on the three variables in the functions $\langle\phi,\theta,\psi|\begin{array}{c} j \\ nm \end{array}\rangle$. In fact, two sets of such operators can be introduced: one acts on the first set (j,m') of discrete indices while the second acts on the second set (j,m) of discrete indices. It is convenient to describe these operators in a *mixed* basis set:

$$\begin{array}{c} \langle\phi,\theta,\psi|\mathcal{J}_z|\begin{array}{c} j \\ m'm \end{array}\rangle^* \\ \swarrow \qquad\qquad \searrow \\ \langle\phi,\theta,\psi|\frac{1}{i}\frac{\partial}{\partial\phi}|\phi',\theta',\psi'\rangle^*\langle\phi',\theta',\psi'|\begin{array}{c} j \\ m'm \end{array}\rangle^* = \langle\phi,\theta,\psi|\begin{array}{c} j \\ n'n \end{array}\rangle^*\langle\begin{array}{c} j \\ n'n \end{array}|\begin{array}{c} j \\ m'm \end{array}\rangle^* m'\delta_{n'm'}\delta_{nm} \end{array} \tag{35}$$

The expression for $\mathcal{J}_z$ in the continuous basis on the left in the bottom line is to be interpreted as

$$\langle\phi,\theta,\psi|\frac{1}{i}\frac{\partial}{\partial\phi}|\phi',\theta',\psi'\rangle = \frac{1}{i}\frac{\partial}{\partial\phi}\delta(\phi'-\phi)\delta(\cos\theta'-\cos\theta)\delta(\psi'-\psi) \tag{36}$$

Operators $\mathcal{J}_\pm = \mathcal{J}_x \pm i\mathcal{J}_y$ can be defined similarly. In the continuous representation they have the differential form

$$\mathcal{J}_\pm \to ie^{\pm i\phi}\left(\cot\theta\frac{\partial}{\partial\phi} \mp i\frac{\partial}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial}{\partial\psi}\right) \tag{37}$$

On the right hand side these shift operators act only on the first index:

$$\mathcal{J}_\pm\langle\phi,\theta,\psi|\begin{array}{c} j \\ mn \end{array}\rangle^* = \langle\phi,\theta,\psi|\begin{array}{c} j \\ m'n' \end{array}\rangle^*\sqrt{(j\mp m)(j\pm m')}\delta_{m',m\pm1}\delta_{n'n} \tag{38}$$

A second dual set of operators $\mathcal{K}_i$ can be introduced that act only on the second index. In continuous coordinates their differential representation is

$$\begin{array}{rcl} \mathcal{K}_z & = & \frac{1}{i}\frac{\partial}{\partial\psi} \\ \mathcal{K}_\pm & = & ie^{\mp i\psi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \mp i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\psi}\right) \end{array} \tag{39}$$

The action of these operators on the Wigner functions is

$$\begin{array}{rcl} \mathcal{K}_z\langle\phi,\theta,\psi| \begin{array}{c} j \\ mn \end{array} \rangle^* & = & \langle\phi,\theta,\psi| \begin{array}{c} j \\ m'n' \end{array} \rangle^* n\delta_{m'm}\delta n'n \\ \mathcal{K}_\pm\langle\phi,\theta,\psi| \begin{array}{c} j \\ mn \end{array} \rangle^* & = & \langle\phi,\theta,\psi| \begin{array}{c} j \\ m'n' \end{array} \rangle^* \delta_{m',m}\sqrt{(j\pm n)(j\mp n')}\delta_{n'n\mp 1} \end{array} \tag{40}$$

The duality between the operators $\mathcal{J}$ and $\mathcal{K}$ is summarized in part by: $\mathcal{J}_+$ shifts the first discrete index up by one while $\mathcal{K}_+$ shifts the second index *down* by one.

The two sets of operators $\mathcal{J}_i$ and $\mathcal{K}_i$ have similar but not identical commutation relations

$$\begin{array}{rclcrcl} [\mathcal{J}_1,\mathcal{J}_2] & = & i\mathcal{J}_3 & \qquad & [\mathcal{K}_1,\mathcal{K}_2] & = & -i\mathcal{K}_3 \\ [\mathcal{J}_2,\mathcal{J}_3] & = & i\mathcal{J}_1 & & [\mathcal{K}_2,\mathcal{K}_3] & = & -i\mathcal{K}_1 \\ [\mathcal{J}_3,\mathcal{J}_1] & = & i\mathcal{J}_2 & & [\mathcal{K}_3,\mathcal{K}_1] & = & -i\mathcal{K}_2 \end{array} \tag{41}$$

In addition, the two sets of operators mutually commute: $[\mathcal{J}_i,\mathcal{K}_j]=0$. Further, the sum of the squares of the two sets of operators

$$\mathcal{J}^2 = \mathcal{J}_1^2+\mathcal{J}_2^2+\mathcal{J}_3^2 \qquad \mathcal{K}^2 = \mathcal{K}_1^2+\mathcal{K}_2^2+\mathcal{K}_3^2 \tag{42}$$

have the same differential representation

$$\mathcal{J}^2 = \mathcal{K}^2 = -\frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial\psi^2}-2\cos\theta\frac{\partial^2}{\partial\phi\partial\psi}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} \tag{43}$$

and have the same eigenvalue $j(j+1)$ in the $2j+1$ finite-dimensional representation [16].

3.7 Relations: Recursion and Differential

The matrix elements of the shift operators $\mathcal{J}_\pm$ can be computed in the mixed basis in two different ways. In the discrete basis we find

$$\begin{aligned}\langle\phi,\theta,\psi|\mathcal{J}_\pm|\begin{matrix}j\\mn\end{matrix}\rangle^* &= \langle\phi,\theta,\psi|\begin{matrix}j\\m'n'\end{matrix}\rangle^*\langle\begin{matrix}j\\m'n'\end{matrix}|\mathcal{J}_\pm|\begin{matrix}j\\mn\end{matrix}\rangle^* \\ &= \langle\phi,\theta,\psi|\begin{matrix}j\\m\pm 1,n\end{matrix}\rangle^*\sqrt{(j\mp m)(j\pm m+1)}\end{aligned} \tag{44}$$

using the results of Eq. (38).

A similar calculation in the continuous basis gives

$$\begin{aligned}\langle\phi,\theta,\psi|\mathcal{J}_\pm|\begin{matrix}j\\mn\end{matrix}\rangle^* &= \langle\phi,\theta,\psi|\mathcal{J}_\pm|\phi',\theta',\psi'\rangle^*\langle\phi',\theta',\psi'|\begin{matrix}j\\mn\end{matrix}\rangle^* \\ &= e^{\pm i\phi}\left(-m\cot\theta\pm\tfrac{\partial}{\partial\theta}+\tfrac{n}{\sin\theta}\right)\langle\phi,\theta,\psi|\begin{matrix}j\\mn\end{matrix}\rangle^*\end{aligned} \tag{45}$$

using the result of Eq. (37) and taking the derivatives with respect to the rotation angles ϕ,ψ around the z-axis.

By taking the linear combination of $e^{-i\phi}\mathcal{J}_+ + e^{+i\phi}\mathcal{J}_-$ we find a recursion relation:

$$\begin{aligned}2\,\frac{n-m\cos\theta}{\sin\theta}\langle\phi,\theta,\psi|\begin{matrix}j\\mn\end{matrix}\rangle^* &= \langle\phi,\theta,\psi|\begin{matrix}j\\m+1,n\end{matrix}\rangle^* e^{-i\phi}\sqrt{(j-m)(j+m+1)} \\ &\quad+\langle\phi,\theta,\psi|\begin{matrix}j\\m-1,n\end{matrix}\rangle^* e^{+i\phi}\sqrt{(j+m)(j-m+1)}\end{aligned} \tag{46}$$

By taking the alternate linear combination $e^{-i\phi}\mathcal{J}_+ - e^{+i\phi}\mathcal{J}_-$ we find a differential relation:

$$\begin{aligned}2\,\frac{\partial}{\partial\theta}\langle\phi,\theta,\psi|\begin{matrix}j\\mn\end{matrix}\rangle^* &= \langle\phi,\theta,\psi|\begin{matrix}j\\m+1,n\end{matrix}\rangle^* e^{-i\phi}\sqrt{(j-m)(j+m+1)} \\ &\quad-\langle\phi,\theta,\psi|\begin{matrix}j\\m-1,n\end{matrix}\rangle^* e^{+i\phi}\sqrt{(j+m)(j-m+1)}\end{aligned} \tag{47}$$

The recursion and differential relations for the first index of the Wigner functions result from computing the matrix elements of the shift operators $\mathcal{J}_\pm$ in the mixed basis in two different ways. Similar results are obtained for the second index by computing the matrix elements of the shift operators $\mathcal{K}_\pm$. The results are

$$2\ \frac{n\cos\theta - m}{\sin\theta}\langle \phi,\theta,\psi| \begin{matrix} j \\ mn \end{matrix} \rangle^* = \langle \phi,\theta,\psi| \begin{matrix} j \\ m,n-1 \end{matrix} \rangle^* e^{+i\psi}\sqrt{(j+n)(j-n+1)} + \langle \phi,\theta,\psi| \begin{matrix} j \\ m,n+1 \end{matrix} \rangle^* e^{-i\psi}\sqrt{(j-m)(j+n+1)} \tag{48}$$

and

$$2\ \frac{\partial}{\partial\theta}\langle \phi,\theta,\psi| \begin{matrix} j \\ mn \end{matrix} \rangle^* = \langle \phi,\theta,\psi| \begin{matrix} j \\ m,n-1 \end{matrix} \rangle^* e^{+i\psi}\sqrt{(j+n)(j-n+1)} - \langle \phi,\theta,\psi| \begin{matrix} j \\ m,n+1 \end{matrix} \rangle^* e^{-i\psi}\sqrt{(j-n)(j+n+1)} \tag{49}$$

These relations have been obtained using the results of Eqs. (39-40).

For the subset of Wigner functions consisting of the spherical harmonics the shift operators $L_\pm$ are

$$L_\pm = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)\delta(\cos\theta' - \cos\theta)\delta(\phi' - \phi) \tag{50}$$

with $L_\pm Y^l_m(\theta,\phi) = Y^l_{m\pm1}(\theta,\phi)\sqrt{(l\mp m)(l\pm m+1)}$. Then the differential relation is

$$2\frac{\partial}{\partial\theta}Y^l_m(\theta,\phi) = e^{-i\phi}\sqrt{(l-m)(l+m+1)}Y^l_{m+1} - e^{+i\phi}\sqrt{(l+m)(l-m+1)}Y^l_{m-1} \tag{51}$$

and the recursion relation is

$$-2m\cot\theta Y^l_m(\theta,\phi) = e^{-i\phi}\sqrt{(l-m)(l+m+1)}Y^l_{m+1} + e^{+i\phi}\sqrt{(l+m)(l-m+1)}Y^l_{m-1} \tag{52}$$

4 Bessel Functions

4.1 Euclidean Group in the Plane

The Euclidean group in the plane, denoted E_2 or $ISO(2)$ (inhomogeneous (I) rotation group (SO) in the plane (2)), consists of all possible translations and rotations in the Euclidean plane R^2. It has a faithful 3×3 matrix representation given by

$$g(x, y, \theta) = \begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \in E_2 \tag{53}$$

with group parameters $(x, y) \in R^2$ and $\theta \in S^1$. The action of this group operation on a point $(a, b) \in R^2$ is given by matrix multiplication on the column vector $\begin{bmatrix} a & b & 1 \end{bmatrix}^t$:

$$\begin{bmatrix} a' \\ b' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} a\cos\theta + b\sin\theta + x \\ -a\sin\theta + b\cos\theta + y \\ 1 \end{bmatrix} \tag{54}$$

The infinitesimal generators for displacements in the x and y directions (A and B) and for rotations about the perpendicular direction (C) are

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{55}$$

The infinitesimal generators in the space of functions defined on the plane are

$$P_x = \frac{\partial}{\partial x} \qquad P_y = \frac{\partial}{\partial y} \qquad L_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{56}$$

These operators satisfy the commutation relations

$$\begin{array}{rclcrcl} [L_z, P_x] & = & -P_y & \qquad & [C, A] & = & -B \\ [L_z, P_y] & = & +P_x & & [C, B] & = & +A \\ [P_x, P_y] & = & 0 & & [A, B] & = & 0 \end{array} \tag{57}$$

Although neither P_x nor P_y commute with L_z, $P_x^2 + P_y^2$ does: $\left[L_z, P_x^2 + P_y^2\right] = 0$. It has the same value on all basis vectors in an irreducible representation.

4.2 Shift Operators

It is useful to define linear combinations of the commuting operators as follows: $P_\pm = +iP_y \pm P_x$. These linear combinations commute and have the following commutation relations with the rotation operator

$$[L_z, P_\pm] = \pm i P_\pm \tag{58}$$

This means that the operators $P_\pm$ can play the role of shift operators.

In the coordinate representation ($x = r\cos\phi, y = r\sin\phi$) these operators have the following differential expressions

$$L_z = \frac{1}{i}\frac{\partial}{\partial\phi} \qquad P_\pm = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi}\right) \tag{59}$$

4.3 Basis Functions

It is possible to choose basis functions to be eigenfunctions of L_z with integer eigenvalues, required by the single-valuedness condition:

$$L_z|n\rangle = n|n\rangle \quad \text{or} \quad \frac{1}{i}\frac{\partial}{\partial\phi}\langle r,\phi|n\rangle = n\langle r,\phi|n\rangle \Rightarrow \langle r,\phi|n\rangle = f_n(r)e^{in\phi} \tag{60}$$

We show now that the radial functions $f_n(r)$ are Bessel functions of integer order. The shift operators $P_\pm$ shift the basis states to the next higher or lower basis states: $P_\pm|n\rangle \simeq |n\pm 1\rangle$. We define adjacent basis states as follows:

$$P_+|n\rangle = p|n+1\rangle \qquad P_-|n\rangle = p|n-1\rangle \tag{61}$$

By applying successively the shift up and down operators, in either order (since they commute), we find

$$P_+P_-\langle r,\phi|n\rangle = P_+p\langle r,\phi|n-1\rangle = p^2\langle r,\phi|n\rangle \tag{62}$$

As a result, p^2 is interpreted as the value of the invariant operator $P_x^2 + P_y^2$. In the coordinate representation the same calculation is

$$e^{+i\phi}\left(+\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi}\right)e^{-i\phi}\left(-\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi}\right)f_n e^{in\phi} =$$

$$e^{+i\phi}\left(+\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\phi}\right)\left(-f_n'(r)-\frac{n}{r}f_n(r)\right)e^{i(n-1)\phi}=\left(-f_n''(r)-\frac{1}{r}f_n'(r)+\frac{n^2}{r^2}f_n(r)\right)e^{in\phi} \tag{63}$$

By equating Eq. (63) with Eq. (62) we find a form of the Bessel equation

$$f_n''(r)+\frac{1}{r}f_n'(r)+\left(p^2-\frac{n^2}{r^2}\right)f_n(r)=0 \tag{64}$$

This reduces to the familiar Bessel equation if we rescale the radial variable $r \to pr$, or equivalently if we choose $p^2 = 1$. To ensure square integrability, these are the Bessel functions of the first kind that have the Taylor series expansion

$$J_n(r)=\sum_{s=0}^{\infty}\left(\frac{r}{2}\right)^n\frac{(-r^2/4)^s}{s!(n+s)!} \tag{65}$$

and the symmetry $J_{-n}(x)=(-1)^nJ_n(x)$.

All real values of p give representations that are unitary. If $p \neq 0$ these representations are irreducible. Representations labeled p and $-p$ are equivalent. We adopt the convention $p=-1$ to conform to the usual phase convention of the Bessel functions, so that

$$\langle r,\phi|n\rangle = J_n(r)e^{in\phi} \tag{66}$$

4.4 Differential and Recursion Relations

With this choice of p,

$$\left(\pm\frac{d}{dr}-\frac{n}{r}\right)J_n(r)=-J_{n\pm1}(r) \tag{67}$$

The mixed basis matrix elements of $P_+ \pm P_-$ are

$$\begin{array}{ccc}
 & \langle r,\phi|P_+\pm P_-|n\rangle & \\
\swarrow & & \searrow \\
\langle r,\phi|P_+\pm P_-|r',\phi'\rangle\langle r',\phi'|n\rangle & = & \langle r,\phi|n'\rangle\langle n'|P_+\pm P_-|n\rangle \\
\downarrow & & \downarrow \\
\left\{\left(+\frac{d}{dr}-\frac{n}{r}\right)\pm\left(-\frac{d}{dr}-\frac{n}{r}\right)\right\}J_n(r) & = & -J_{n+1}(r)\mp J_{n-1}(r)
\end{array} \tag{68}$$

The positive sign provides the recursion relation and the negative sign the differential relation

$$\begin{array}{rrcl} -P_+ - P_- \Rightarrow & \frac{2n}{r} J_n(r) & = & J_{n-1}(r) + J_{n+1}(r) \\ +P_+ - P_- \Rightarrow & 2\,\frac{d}{dr} J_n(r) & = & J_{n-1}(r) - J_{n+1}(r) \end{array} \tag{69}$$

4.5 Generating Function

In the unitary representation with $p = -1$ the rotation operator is diagonal: $\langle n'|e^{iL_z\phi}|n\rangle = e^{in\phi}\delta_{n'n}$. The shift operators are more complicated:

$$\langle n'|e^{xP_x+yP_y}|n\rangle = \langle n'|e^{aP_+ + bP_-}|n\rangle \qquad a = (r/2)e^{-i\phi}, b = (-r/2)e^{+i\phi} \tag{70}$$

The matrix representatives of the shift operators $P_\pm$ have nonzero values -1 only on the diagonal above (for P_+) or below (for P_-) the main diagonal. Further, both matrices are cyclic: each row is the same as any other, centered on the diagonal. As a result e^{aP_+} is upper triangular and cyclic, e^{bP_-} is lower triangular and cyclic, and their product is cyclic and has matrix elements

$$\left(e^{bP_-}e^{aP_+}\right)_{p+n,p} = \sum_{s=0} \frac{(-b)^n (ba)^s}{s!(n+s)!} \xrightarrow{(a,b)\to(r,\phi)} e^{in\phi} \sum_{s=0} \left(\frac{r}{2}\right)^n \frac{(-r^2/4)^s}{s!(s+n)!} = e^{in\phi} J_n(r) \tag{71}$$

This holds for n negative by the index symmetry expressed below Eq. (65). In summary, the matrix elements of a translation are

$$\langle n'|e^{xP_x+yP_y}|n\rangle = e^{i(n'-n)\phi} J_{n'-n}(r) \tag{72}$$

In this respect, with $z = e^{i\phi}$ the function $e^{(r/2)(z-z^{-1})}$ can be considered as the generating function for the matrix elements in any row of the representation of the translation operator:

$$e^{(r/2)(z-z^{-1})} = \sum_n z^n J_n(r) \tag{73}$$

4.6 Addition Theorem

The addition theorems arise from expressions such as

$$\begin{aligned}
\langle n'|e^{x_1P_x+y_1P_y}\ e^{x_2P_x+y_2P_y}|n\rangle &= \langle n'|e^{(x_1+x_2)P_x+(y_1+y_2)P_y}|n\rangle \\
\sum_p \langle n'|e^{x_1P_x+y_1P_y}|p\rangle\langle p|e^{x_2P_x+y_2P_y}|n\rangle &= e^{i(n'-n)\Phi}J_{n'-n}(R) \qquad (74) \\
\sum_p e^{i(n'-p)\phi_1}J_{n'-p}(r_1)e^{i(p-n)\phi_2}J_{p-n}(r_2) &= e^{i(n'-n)\Phi}J_{n'-n}(R)
\end{aligned}$$

To be explicit

$$\begin{aligned}
r_1\cos\phi_1 + r_2\cos\phi_2 &= R\cos\Phi \\
r_1\sin\phi_1 + r_2\sin\phi_2 &= R\sin\Phi
\end{aligned} \qquad (75)$$

5 Heisenberg Groups: H_4 and H_3

The Heisenberg algebra $\mathfrak{h}_4$ is spanned by the number, creation, annihilation, and identity operators $\hat{n}, a^\dagger, a, I$. The Heisenberg group H_4 is obtained by exponentiation these operators. These operators satisfy the well-known commutation relations

$$\left[\hat{n}, a^\dagger\right] = +a^\dagger \qquad [\hat{n}, a] = -a \qquad \left[a, a^\dagger\right] = I \qquad [I, *] = 0 \qquad (76)$$

The Heisenberg algebra $\mathfrak{h}_3$ is the subagebra spanned by the creation, annihilation, and identity operators, and H_3 is the corresponding Heisenberg Group.

The creation and annihilation operators are obtained by "factoring" the harmonic oscillator hamiltonian into a pair of linear factors in the spirit of Dirac's factorization of the Klein-Gordan equation into a pair of linear factors:

$$\frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right) \simeq \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) \times \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right) \qquad (77)$$

This product does not exactly equal the hamiltonian (with $\hbar = m = k = 1$) on the left. When the linear factors are multiplied in the reverse order and added to the product above and the result is averaged, the hamiltonian results. The following identifications are standard:

$$a = \frac{1}{\sqrt{2}}(x + D) \quad a^\dagger = \frac{1}{\sqrt{2}}(x - D) \tag{78}$$

with $D \equiv \frac{d}{dx}$.

5.1 Faithful 3×3 Matrix Representation

The number, creation, annihilation, and identity operators that span $\mathfrak{h}_4$ can be mapped into a faithful 3×3 matrix representation as follows:

$$\eta\hat{n} + ra^\dagger + la + \delta I \rightarrow \begin{bmatrix} 0 & l & \delta \\ 0 & \eta & r \\ 0 & 0 & 0 \end{bmatrix} \tag{79}$$

This representation is not hermitian, so is not directly applicable to problems of a quantum mechanical nature. In addition, the identity operator is not a multiple of the unit matrix. This is not a problem: the only important property is that the matrix representative of the identity matrix commutes with all other operators in this algebra.

The virtue of this matrix representation is that it is easily exponentiated

$$EXP \begin{bmatrix} 0 & l & \delta \\ 0 & \eta & r \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & l\frac{e^\eta - 1}{\eta} & \delta + lr\frac{e^\eta - 1 - \eta}{\eta^2} \\ 0 & e^\eta & r\frac{e^\eta - 1}{\eta} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\eta \rightarrow 0} \begin{bmatrix} 1 & l & \delta + \frac{lr}{2} \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} \tag{80}$$

Simple matrix multiplication using the exponentials above lead to the following useful distentangling theorems:

$$e^{ra^\dagger + la} = e^{ra^\dagger} e^{rl/2} e^{la} = e^{la} e^{-rl/2} e^{ra^\dagger} \tag{81}$$

The operator on the left is unitary provided $(ra^\dagger + la)^\dagger = -(ra^\dagger + la)$, or $l = -r^*$.

5.2 Discrete Basis

It is useful to introduce a discrete set of basis states $|n\rangle$, $n = 0, 1, 2, \cdots$. The matrix elements of the four operators in this orthonormal basis $\langle n'|n\rangle = \delta_{n',n}$

are

$$\begin{array}{rclrcl} \langle n'|\hat{n}|n\rangle & = & (n+\frac{1}{2})\delta_{n',n} & \langle n'|a^\dagger|n\rangle & = & \sqrt{n'}\delta_{n',n+1} \\ \langle n'|I|n\rangle & = & 1\delta_{n',n} & \langle n'|a|n\rangle & = & \sqrt{n}\delta_{n',n-1} \end{array} \tag{82}$$

As with the set of unitary representations of the Euclidean group $E(2)$ described by the real number p, there are many inequivalent unitary representations of the Heisenberg group H_4 described by different values of a real number. We have chosen the representation that occurs in most physical applications.

The basis state $|n\rangle$ can be obtained by the action of the creation operator on the "ground state" $|0\rangle$ by repeated application:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{83}$$

The operator a annihilates the ground state: $a|0\rangle = 0$.

5.3 Harmonic Oscillator Wavefunctions

The matrix elements of the creation and annihilation operators in a mixed basis provide the harmonic oscillator eigenfunctions. The ground state is determined from

$$\begin{array}{ccc} & \langle x|a|0\rangle & \\ \swarrow & & \searrow \\ \langle x|a|x'\rangle\langle x'|0\rangle & = & \langle x|n'\rangle\langle n'|a|0\rangle \\ \downarrow & & \downarrow \\ \frac{1}{\sqrt{2}}\left(x+\frac{d}{dx}\right)\langle x|0\rangle & = & 0 \end{array} \tag{84}$$

This linear ordinary differential equation has a unique solution (up to sign) which, when normalized to unity, is

$$\psi_0(x) = \langle x|0\rangle = \frac{1}{\sqrt[4]{\pi}}e^{-x^2/2} \tag{85}$$

The functions $\langle x|n\rangle$ are computed in a straightforward way:

$$
\begin{array}{ccc}
 & \langle x|n\rangle = \langle x|\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle & \\
\swarrow & & \searrow \\
\langle x|n'\rangle\langle n'|\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle & = & \langle x|\frac{(a^\dagger)^n}{\sqrt{n!}}|x'\rangle\langle x'|0\rangle \\
\langle x|n\rangle & = & \frac{(x-D)^n}{\sqrt{2^n n!\sqrt{\pi}}}e^{-x^2/2} \\
\psi_n(x) & = & \frac{H_n(x)e^{-x^2/2}}{\sqrt{2^n n!\sqrt{\pi}}}
\end{array}
\tag{86}
$$

Here $\psi_n(x)$ are the standard normalized harmonic oscillator eigenfunctions and $H_n(x)$ the standard Hermite polynomials.

5.4 Differential and Recursion Relations

The operators $a \pm a^\dagger$ can be computed in a mixed basis in the usual way. The two sign choices give the recursion and differential relations:

$$
\begin{array}{ccc}
 & \langle x|a \pm a^\dagger|n\rangle & \\
\swarrow & & \searrow \\
\langle x|a \pm a^\dagger|x'\rangle\langle x'|n\rangle & = & \langle x|n'\rangle\langle n'|a \pm a^\dagger|n\rangle \\
 & & \langle x|n-1\rangle\sqrt{n} \pm \langle x|n+1\rangle\sqrt{n+1}
\end{array}
\tag{87}
$$

Choice of the positive sign gives the recursion relation and the negative sign gives the differential relation:

$$
\begin{array}{lrcl}
a + a^\dagger \rightarrow & \sqrt{2}x\psi_n(x) & = & \sqrt{n}\psi_{n-1}(x) + \sqrt{n+1}\psi_{n+1}(x) \\
a - a^\dagger \rightarrow & \sqrt{2}\frac{d}{dx}\psi_n(x) & = & \sqrt{n}\psi_{n-1}(x) - \sqrt{n+1}\psi_{n+1}(x)
\end{array}
\tag{88}
$$

5.5 Generating Function

A generating function is obtained by constructing two equivalent representations for the group operator $e^{t\sqrt{2}(a+a^\dagger)}$ using disentangling theorems Eq. (81)

$$
e^{t\sqrt{2}(a^\dagger+a)} = e^{t\sqrt{2}a^\dagger}e^{t\sqrt{2}a}e^{t^2 I} = e^{2tx} \tag{89}
$$

and then applying these expressions to the ground state $\langle x|0\rangle$:

$$
\begin{array}{rcl}
 & \langle x|e^{t\sqrt{2}(a^\dagger+a)}|0\rangle & \\
\langle x|e^{t\sqrt{2}(a^\dagger+a)}|x'\rangle\langle x|0\rangle & = & \langle x|n\rangle\langle n|e^{t\sqrt{2}(a^\dagger+a)}|0\rangle \\
e^{2tx}\langle x|0\rangle & = & \langle x|n\rangle\langle n|e^{t\sqrt{2}a^\dagger}e^{\sqrt{2}ta}e^{t^2}|0\rangle \\
e^{2tx}e^{-x^2/2}/\sqrt[4]{\pi} & = & \langle x|\frac{(\sqrt{2}t)^n}{\sqrt{n!}}|n\rangle e^{t^2} \\
e^{2tx-t^2} & = & \sum_{n=0}\frac{t^n H_n(x)}{n!}
\end{array}
\tag{90}
$$

5.6 Addition Formulas

Addition theorems are most easily constructed from the generating functions.

$$
\begin{array}{rcl}
e^{2tx_1-t^2}e^{2tx_2-t^2} & = & e^{2t(x_1+x_2)-2t^2} = e^{2s((x_1+x_2)/\sqrt{2})-s^2} \\
\sum_p\sum_q \frac{t^p H_p(x_1)}{p!}\frac{t^q H_q(x_2)}{q!} & = & \sum_r \frac{(t\sqrt{2})^r H_r((x_1+x_2)/\sqrt{2})}{r!}
\end{array}
\tag{91}
$$

where $s = t\sqrt{2}$. By equating equal powers of the parameter t on both sides we find

$$
H_r(\frac{(x_1+x_2)}{\sqrt{2}}) = 2^{-r/2}\sum_s \frac{r!}{s!(r-s)!}H_s(x_1)H_{r-s}(x_2) \tag{92}
$$

5.7 Generating Function for Unitary Matrix Elements

A simple generating function for discrete basis vectors can be constructed by applying a unitary transformation $U(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ to the ground state

$$
e^{\alpha a^\dagger-\alpha^* a}|0\rangle = e^{\alpha a^\dagger}e^{-\alpha\alpha^*/2}e^{-\alpha^* a}|0\rangle = e^{\alpha a^\dagger}|0\rangle e^{-\alpha^*\alpha/2} = \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle e^{-\alpha^*\alpha/2} \tag{93}
$$

The generating function for the dual states $\langle n|$ is also immediate

$$\left(e^{\alpha a^\dagger - \alpha^* a}|0\rangle\right)^\dagger = \langle 0|e^{-(\alpha a^\dagger - \alpha^* a)} = \sum_n \langle n|\frac{(\alpha^*)^n}{\sqrt{n!}}e^{-\alpha^*\alpha/2} \tag{94}$$

A generating function for the matrix elements of $U(\beta) = e^{\beta a^\dagger - \beta^* a}$ is obtained by computing the ground state expectation value

$$\begin{array}{ccc} & \langle 0|U(-\alpha)U(\beta)U(\alpha)|0\rangle & \\ \swarrow & & \searrow \\ \sum_p \sum_q \frac{(\alpha^*)^p}{\sqrt{p!}}\langle p|U(\beta)|q\rangle \frac{(\alpha)^q}{\sqrt{q!}}e^{-\alpha^*\alpha} & = & e^{\alpha^*\beta - \alpha\beta^* - \beta^*\beta/2} \end{array} \tag{95}$$

From this it is easy to compute the desired matrix element as an infinite sum:

$$\langle p|U(\beta)|q\rangle = \sum_{r=0}^{\min(p,q)} \frac{\sqrt{p!}\sqrt{q!}}{r!}\frac{(\beta)^{p-r}(-\beta^*)^{q-r}}{(p-r)!(q-r)!}e^{-\beta^*\beta/2} \tag{96}$$

$$\langle p|U(\beta)|q\rangle = \frac{(\beta)^p(-\beta^*)^q}{\sqrt{p!}\sqrt{q!}}\ e^{-\beta^*\beta/2}\ {}_2F_0(-p,-q;-1/\beta^*\beta) \tag{97}$$

In this last expression ${}_2F_0$ is a function of hypergeometric type.

6 Discussion and History

Special functions have had a long history on the Mathematics side of the Mathematical Physics domain. They entered the Physics side in the form of solutions to (usually) second order ordinary differential equations of interest to natural philosophers. This is evident in their canonical names: most are identified with eighteenth and nineteenth century scientists. The formulation of an encompassing theory of special functions has had many incarnations.

A "modern" era of special functions can be associated with the development of modern quantum mechanics in its Wave Mechanics formulation by Schrödinger in 1926. In the early stages of the development of this theory almost all the special functions were encountered. Inspired by Dirac's factorization of the Klein-Gordon equation into two linear factors [17], noncommuting if the electromagnetic field is present, Schrödinger wondered if factorization of second order linear ordinary differential equations was generally possible. He showed that it was in 1940-1941 [18, 19, 20]. This set the stage for Infeld and Hull to formulate the theory of special functions in terms of a limited number of types of factorization (types A through F) [21]. All factors in each type had a similar structure: $a(x)\frac{d}{dx}+b(x)$. It didn't take long to appreciate that the infinitesimal generators of Lie groups, acting on suitable spaces, have this same structure. In 1964 Willard Miller, Jr. published a short AMS study setting out this connection in one way [10]. In 1965 N. Ja. Vilenkin published his formulation, somewhat different than Miller's, in a Russian text. Concurrently, Eugene Wigner had been teaching a course on this subject at Princeton University. After these beginnings, 1968 was a banner year for this subject. In that year Miller published the expansion of his 43 page Memoir in full length book form [11]; Wigner's lecture notes were transcribed and published with Wigner's permission by Talman [12]; and Vilenkin's book was translated and published in English [13]. Since then the connection between Lie groups and the classical functions of mathematical physics has been known to advanced-level researchers.

We hope that this short formulation of material that is basically well-known at advanced levels will bring an understanding of this connection to the undergraduate level.

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