A Chaotic Walk with Friends

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Birthday Party
CORIA, France

June 23, 2011

Thank You to All My Friends

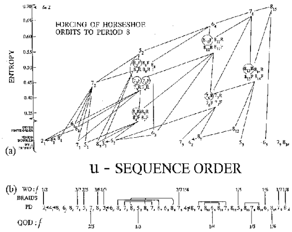
My colleagues and my friends — my colleagues *are* my friends — introduced me to and then helped to guide me through this new and delightful field.

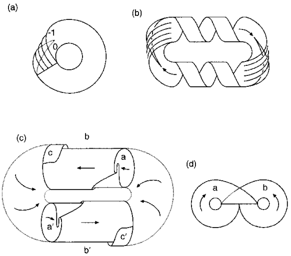
Many are assembled here today.

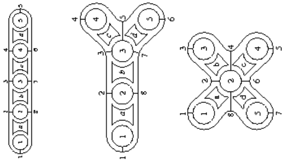
To all I express my thanks for helping to make this such a festive occassion.

Outline

Things to Come







Where it all began:

Hey Bob, these guys down the hall are doing fantastic stuff ...

Elia Eschenazi
The go-between

Usual Culprits: 2

Where it all began:

My friends tell me you like to speak to experimentalists ...

Jorge Tredicce
"I like to ski the walls ...."

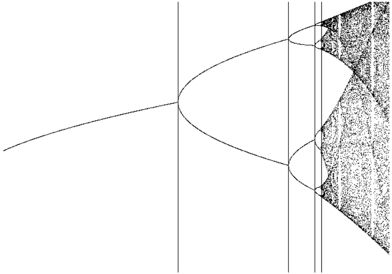
Torreador

J. R. Tredicce

Can you explain my data?

I bet you can't explain my data!

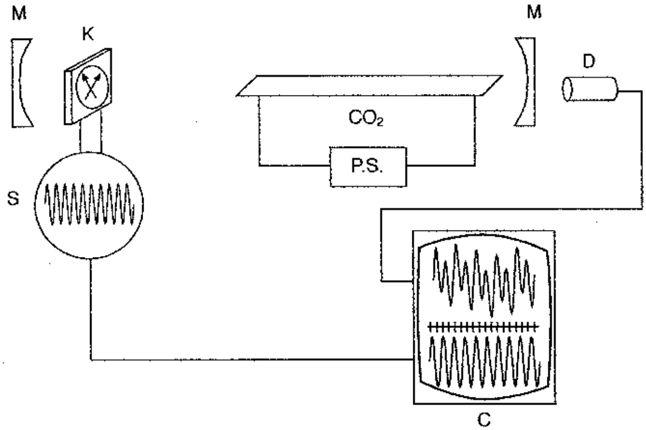
Where is Tredicce coming from?



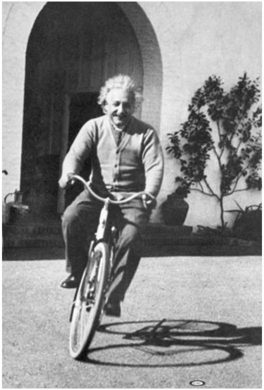
Feigenbaum:

$$\alpha = \quad 4.66920\ 16091\ \ldots\ldots$$
$$\delta = -2.50290\ 78750\ \ldots\ldots$$

Laser with Modulated Losses Experimental Arrangement



What to Grab Hold of ??



Search for Invariants

How to Characterize a Strange Attractor

The Ω limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

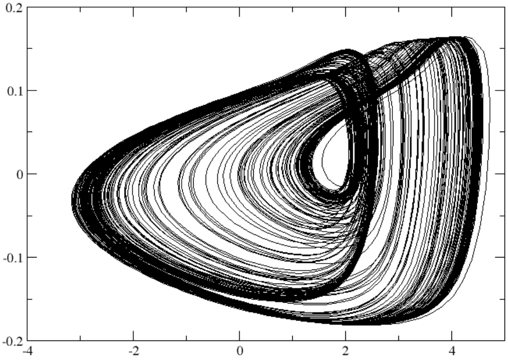
- **"Abundant"**
- **Outline the Strange Attractor**
- **Are the Skeleton of the Strange Attractor**

Periodic Orbits are the Key

Joseph Fourier
Linear Systems

Henri Poincare
Nonlinear Systems

UPOs: Skeletons of Strange Attractors



Here is a Strange Attractor (Belousov-Zhabotinskii Reaction)

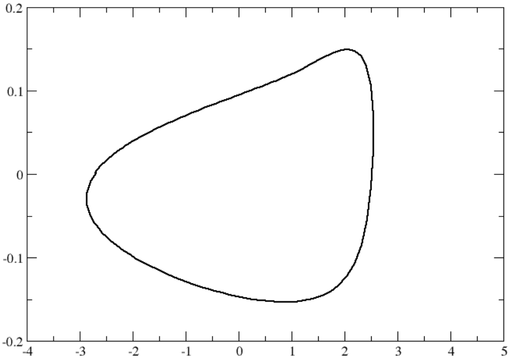
Usual Culprits: 3 & 4

The Topological Team

Hernan G. Solari
(disguised as Cristal)

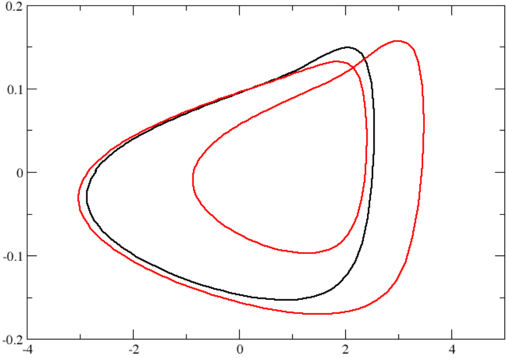
Gabriel B. Mindlin

UPOs: Skeletons of Strange Attractors



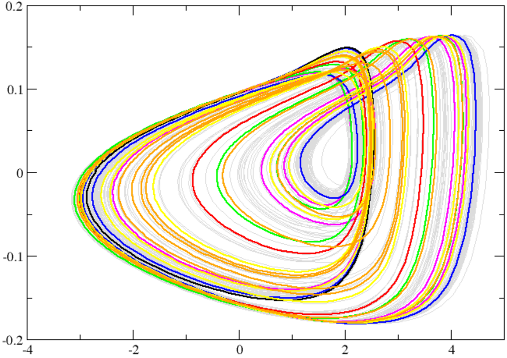
Here is a period-one orbit in the attractor.

UPOs: Skeletons of Strange Attractors



Period-1 and period-2 orbits from the attractor.

UPOs: Skeletons of Strange Attractors



Lots of them.

Ask the Masters: 4

Quantitative Measures for Periodic Orbits ??

Carl Friedrich Gauss

They Link: Pairwise, 3-Wise, ...

Dynamics and Topology

Organization of UPOs in R^3:

Gauss Linking Number

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

$\#$ Interpretations of LN $\simeq$ $\#$ Mathematicians in World

Linking Number of Two UPOs

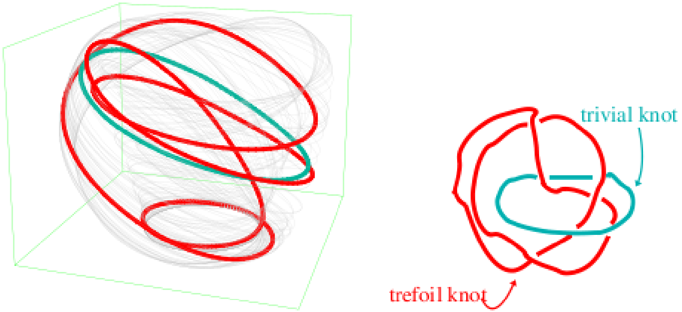


Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

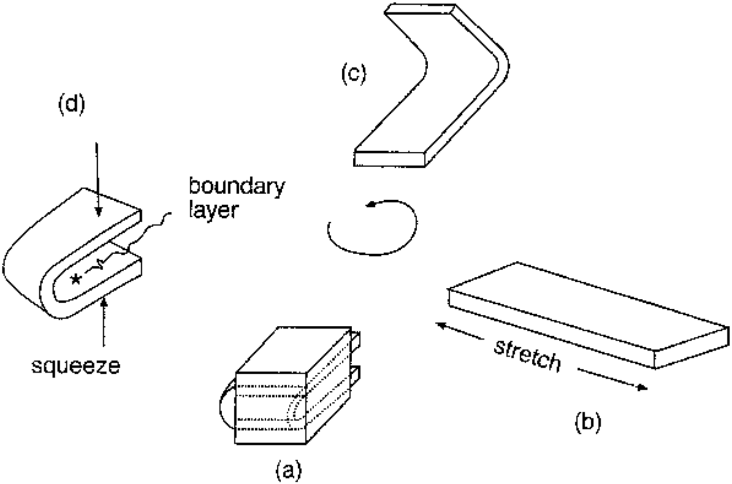
Compute Table of Experimental LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data[a]

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

Stretching and Folding



Mechanisms for Generating Chaos

A Chaotic Walk with Friends

Robert Gilmore

Introduction-01

Introduction-02

Introduction-03

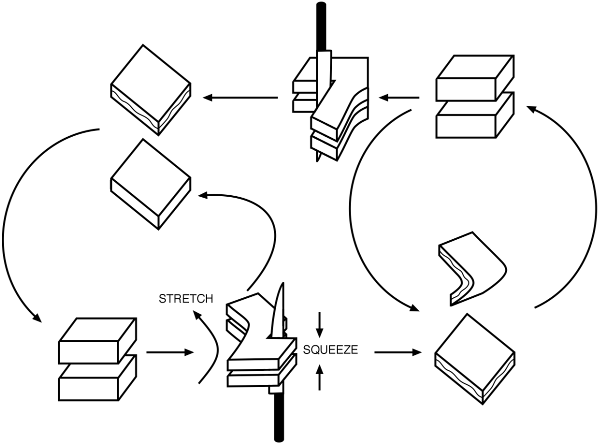
Deep Background-01

Deep Background-02

Deep Background-03

Deep Background-04

Tearing and Squeezing



Ask the Masters: 5 & 6

Systematics of Linking Number Tables

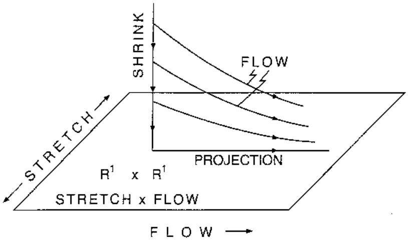
Joan S. Birman — Robert F. Williams

Construct a Branched Manifold

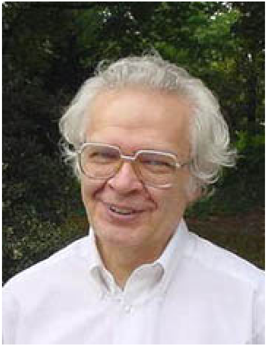
Birman - Williams Projection

Identify x and y if

$$\lim_{t\to\infty} |x(t) - y(t)| \to 0$$



Two Standard Strange Attractors

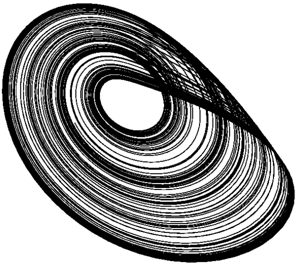


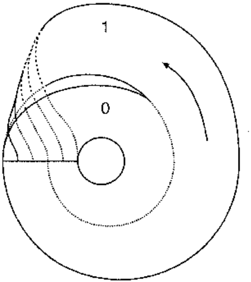
Otto Rössler

Edward Lorenz

Rössler:

Attractor **Branched Manifold**

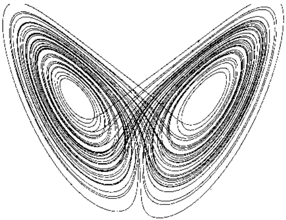


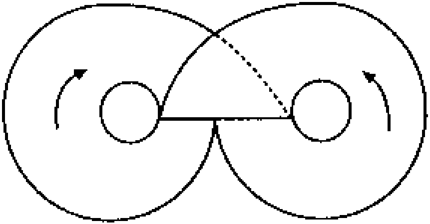


A Mechanism with Symmetry

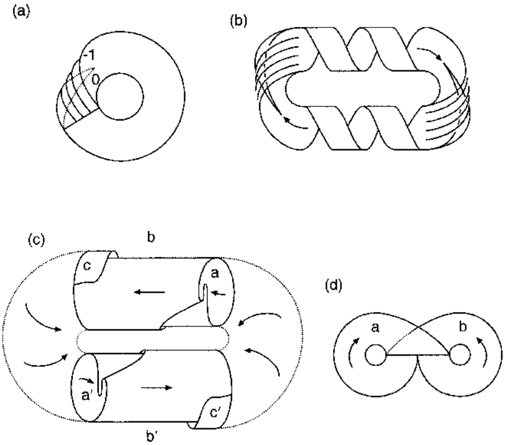
Lorenz:

Attractor **Branched Manifold**

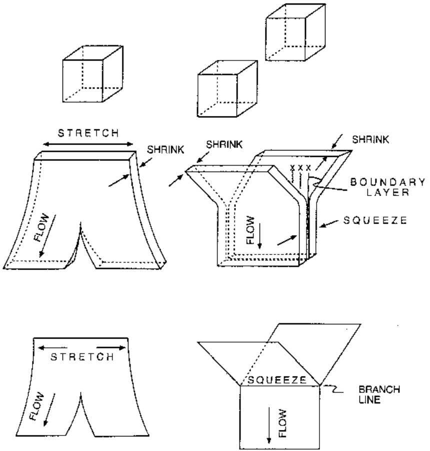




Inequivalent Branched Manifolds

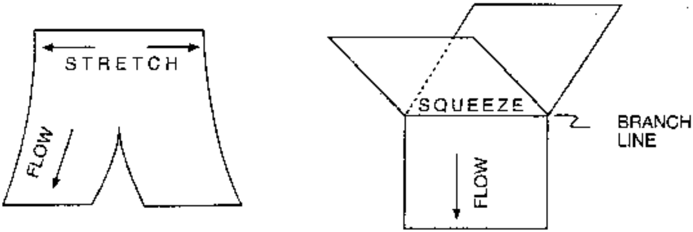


Stretching — Squeezing



Aufbau Princip for Branched Manifolds

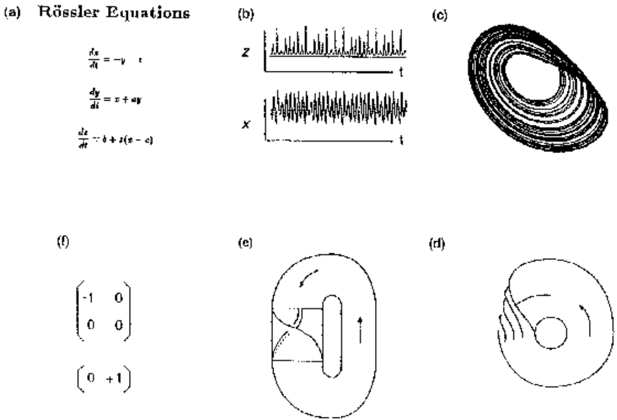
Any branched manifold can be built up from stretching and squeezing units



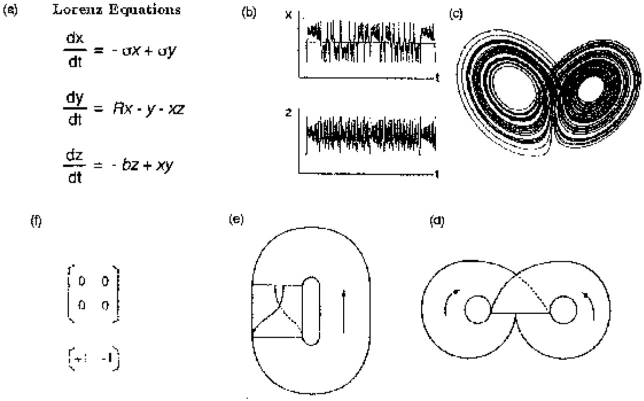
subject to the conditions:

- Outputs to Inputs
- No Free Ends

Rössler System



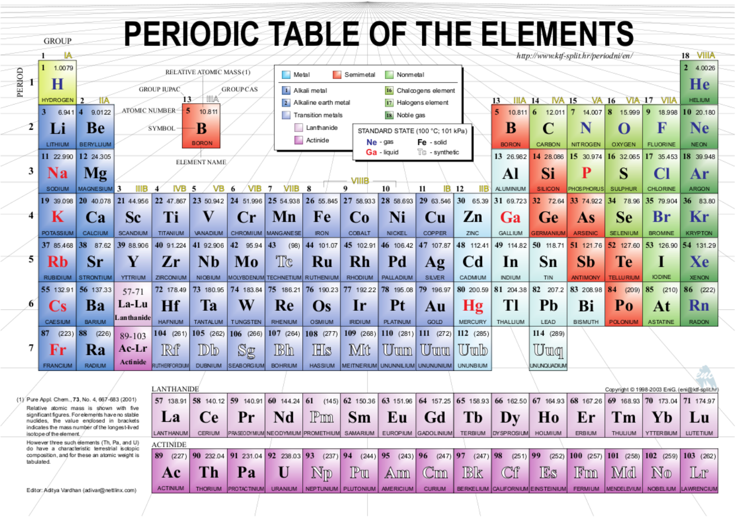
Lorenz System



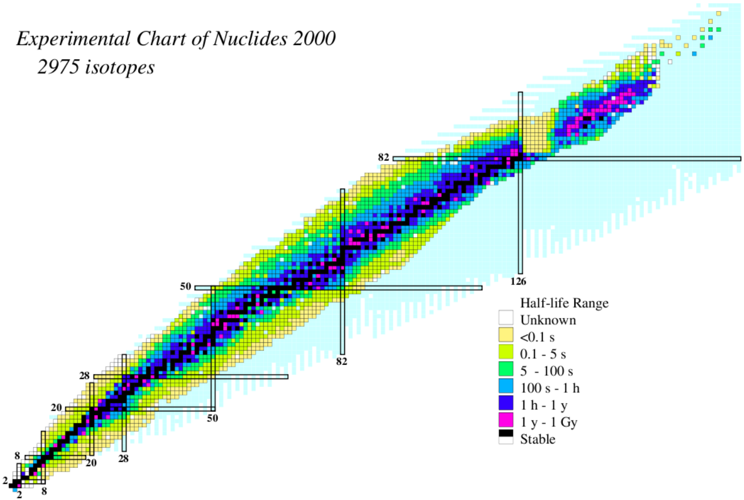
Poincaré Smiles at Us in R^3

- **Determine organization of UPOs** $\Rightarrow$
- **Determine branched manifold** $\Rightarrow$
- **Determine equivalence class of** $\mathcal{SA}$

We Like to be Organized



We Like to be Organized



Topological Analysis Program

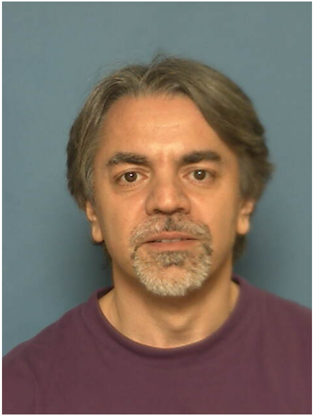
- Locate Periodic Orbits
- Create an Embedding
- Determine Topological Invariants (LN)
- Identify a Branched Manifold
- Verify the Branched Manifold

Additional Steps

- Model the Dynamics
- Validate the Model

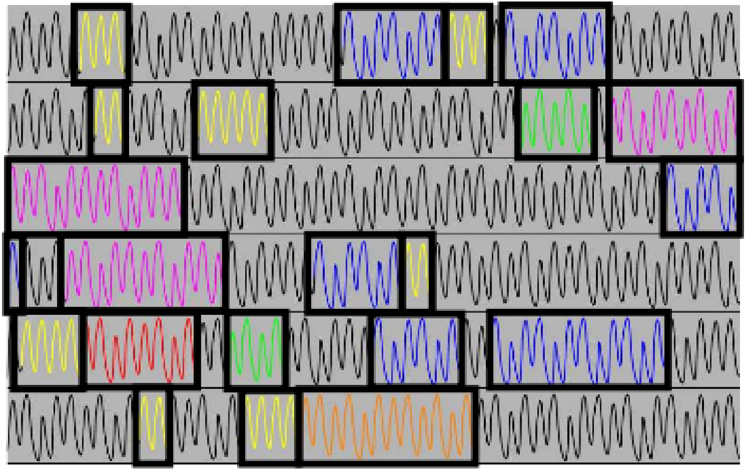
More Team Members

Nicholas B. Tufillaro



Mario A. Natiello

Method of Close Returns



Embeddings (= Black Magic)

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological†

None Good

We Demand a 3 Dimensional Embedding

An Embedding and Periodic Orbits

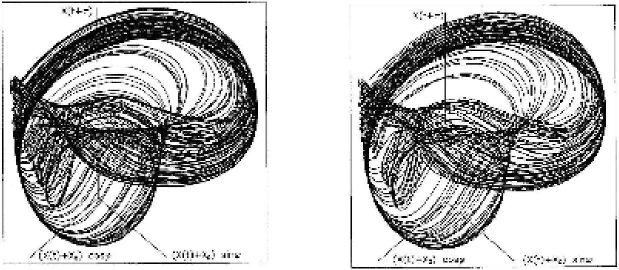


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese

The Topology of Chaos

Marc Lefranc

Compute Table of Experimental LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data[a]

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
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4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

Compare Topological Invariants

Compare w. LN From Various $\mathcal{BM}$

Table 2.1 Linking numbers for orbits to period five in Smale horseshoe dynamics.

	$1s$	$1f$	2_1	$3f$	$3s$	4_1	4_2f	4_2s	5_3f	5_3s	5_2f	5_2s	5_1f	5_1s
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

Lefranc - Cargese

Guess Branched Manifold

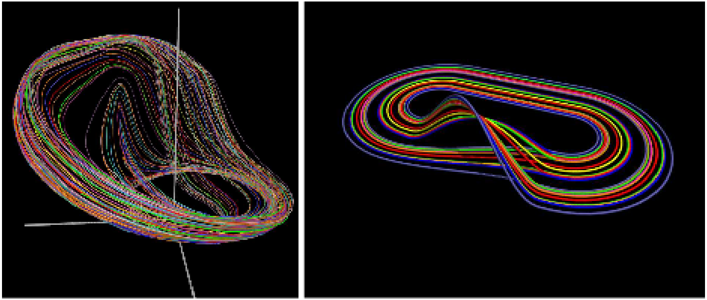


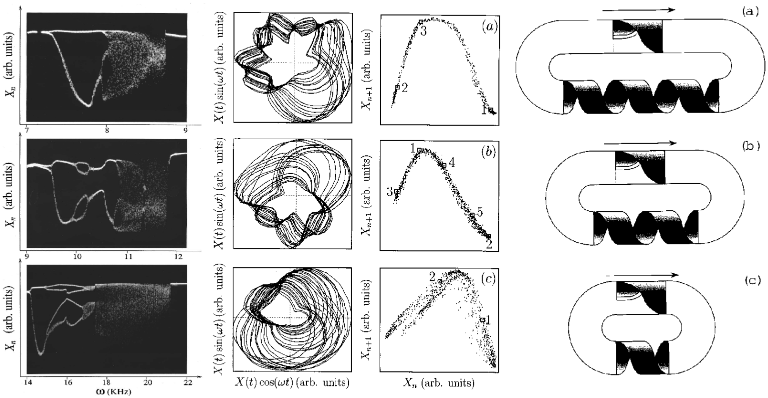
Figure 7. "Combing" the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

Lefranc - Cargese

Identification & 'Confirmation'

- $\mathcal{BM}$ Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion (Fail to Reject H_0)

Evolution Under Parameter Change



Evolution Under Parameter Change

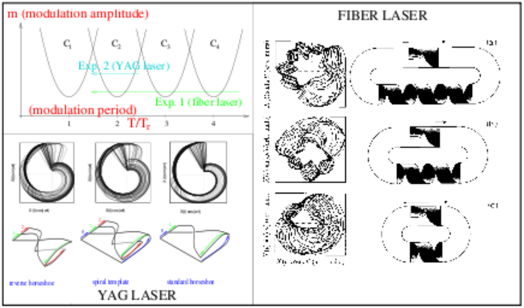


Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

Analysis of Nonstationary Data

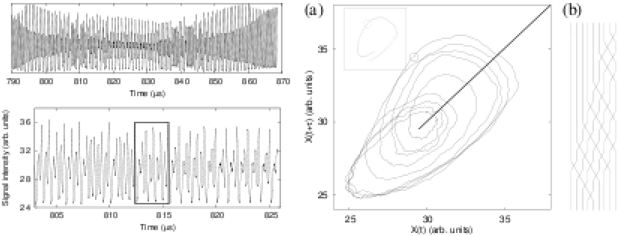


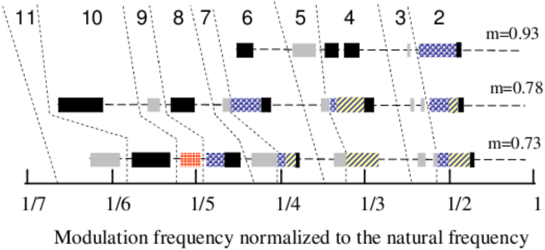
Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

Lefranc - Cargese

A Prediction Realized

TABLE 1 – Folding processes characteristic of the different species of templates treated in this work

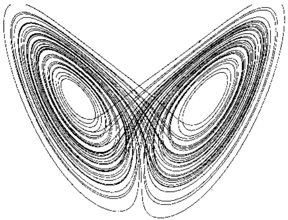
Species	Horseshoe	Reverse horseshoe	Out-to-in spiral	In-to-out spiral	Staple	S
Code in Fig. 1				Not found here		
Insertion matrix	(0 1)	(1 0)	(0 2 1)	(1 2 0)	(0 2 1) or (1 2 0)	(2 1 0)
Sketch of the folding process						



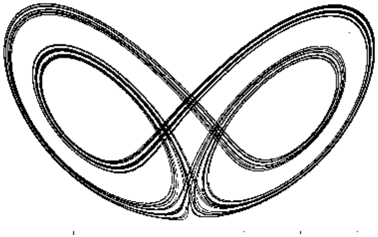
Used and Martin (2010)

Orbits Can be "Pruned"

There Are Some Missing Orbits



Lorenz



Shimizu-Morioka

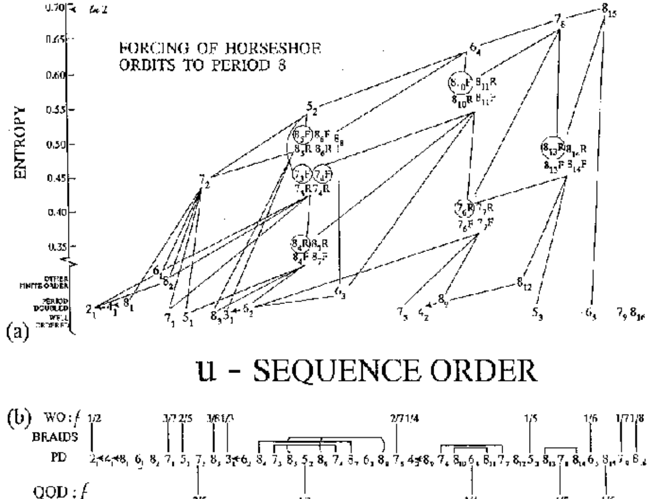
There Are Some Missing Orbits

Francisco Papoff

Arimondo et al.

- The branched manifold remains unchanged,
- the spectrum of orbits on it changes.

Forcing Diagram - Horseshoe

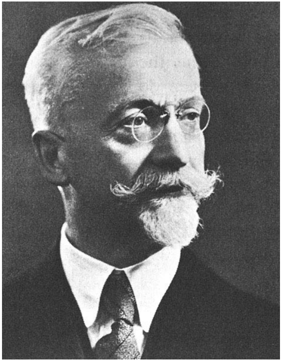


Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required

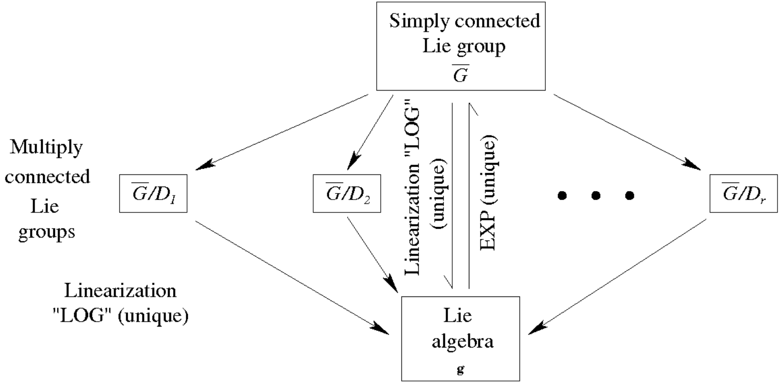
Ask the Masters: 9

What is the relation between
symmetry groups & strange attractors?



Elie Cartan

Cartan's Theorem for Lie Groups

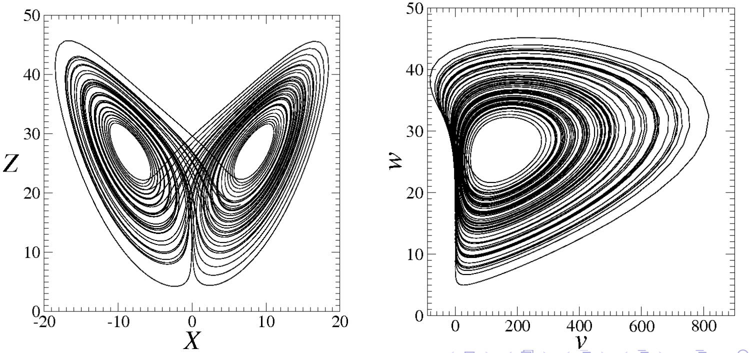


The Symmetry of Chaos

Christophe Letellier

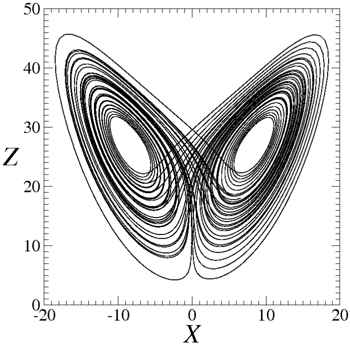
Modding Out a Rotation Symmetry

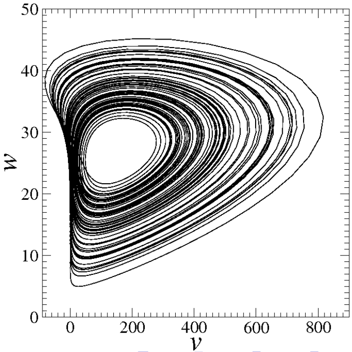
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Re\ (X+iY)^2 \\ Im\ (X+iY)^2 \\ Z \end{pmatrix}$$



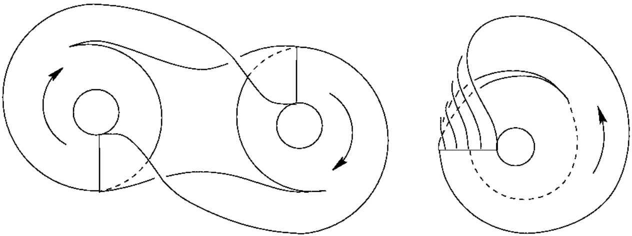
Creating a Cover with Symmetry

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \leftarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Re\ (X+iY)^2 \\ Im\ (X+iY)^2 \\ Z \end{pmatrix}$$





Cover-Image Branched Manifolds



Constraints on Branched Manifolds

- **"Inflate" a strange attractor**
- **Union of ϵ ball around each point**
- **Boundary is surface of bounded 3D manifold**
- **Torus that bounds strange attractor**

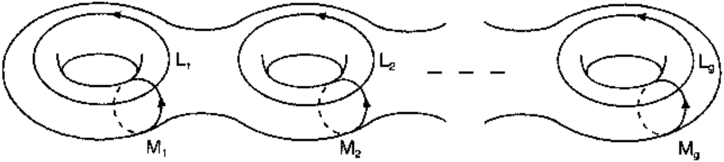
Ask the Masters: 10

How do we characterize surfaces?

Leonard Euler

Count holes: $\chi(\partial\mathcal{M})$

Torus, Longitudes, Meridians



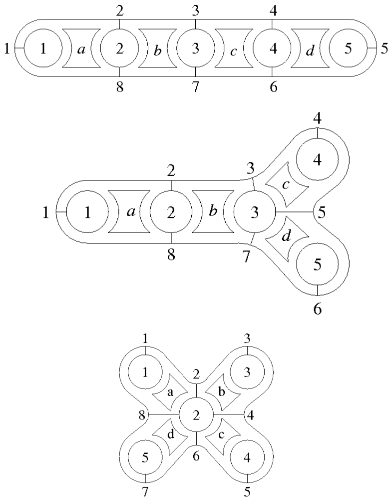
Usual Culprits: 10

What kind of “dressed tori” enclose strange attractors?

Tsvetelin D. Tsankov

Markov Matrices and Symmetric Cycles

Some Genus-9 Bounding Tori



Labeling Bounding Tori

Poincaré section is disjoint union of g-1 disks

Transition matrix sum of two g-1 $\times$ g-1 matrices

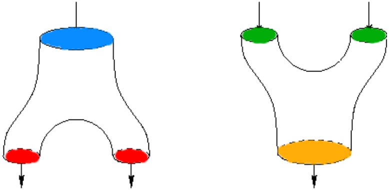
One is cyclic g-1 $\times$ g-1 matrix

Other represents union of cycles

Labeling via (permutation) group theory

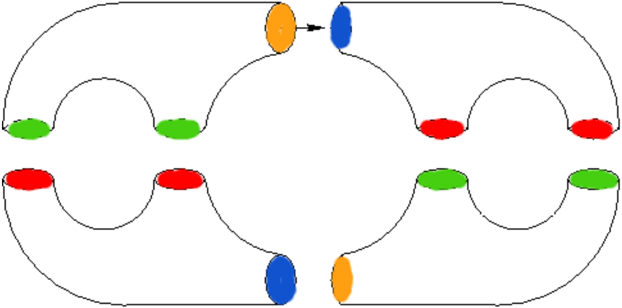
Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units



- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

Application: Lorenz Dynamics, g=3



Construction of Poincaré Section

P. S. = Union 

Components = g-1

The Growth is Exponential

TABLE I: Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

TABLE I: Enumeration of canonical forms up to genus 9

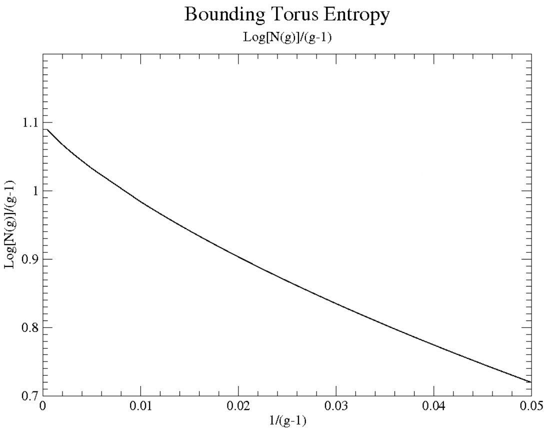
g	m	$(p_1, p_2, \ldots p_m)$	$n_1 n_2 \ldots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11311313
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

Usual Culprits: 11

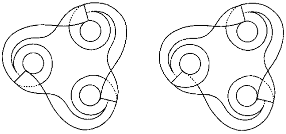
How quickly does the number of bounding tori increase with g?

Jacob Katriel
Magician with permutation group cycles.

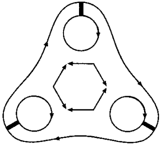
The Growth is Exponential
The Entropy is log 3



Two possible branched manifolds in the torus with g=4.

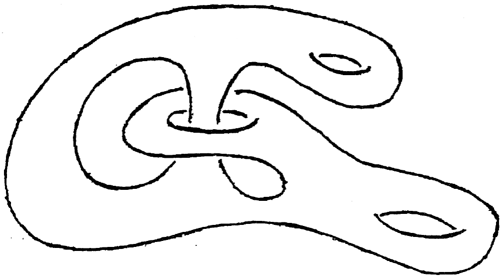


(a) (b)



(c)

Extrinsic Embedding of Intrinsic Tori



Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.

Ask the Masters: 11

Could there be representation theory for strange attractors?

Eugene Wigner

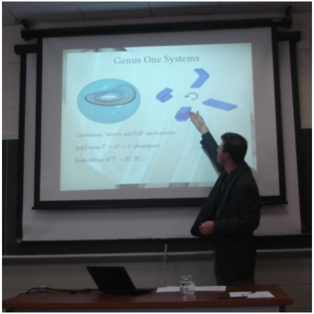
Of course! There is a representation theory for everything!

Usual Culprits: 12

There is a representation theory for strange attractors!

What is it? How does it work?

Daniel J. Cross
Hard at work (pretending)!



Daniel J. Cross
Instructing us.

Usual Culprits: 13 & 14

After fixed points — Organizing curves?

Tim Jones
What?

Jean-Marc Ginoux
How ?

Summary

1 Question Answered $\Rightarrow$

2 Questions Raised

We must be on the right track !

Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Result

There is now a classification theory for low-dimensional strange attractors.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to R^3 only — for now

The Classification Theory has 4 Levels of Structure

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1. Basis Sets of Orbits

The Classification Theory has 4 Levels of Structure

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2. Branched Manifolds

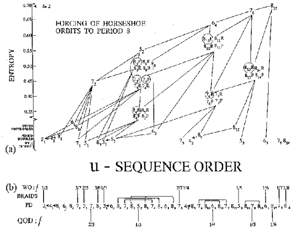
The Classification Theory has 4 Levels of Structure

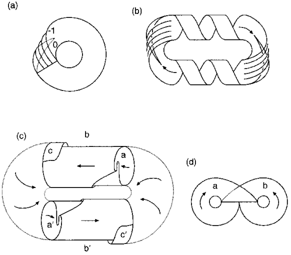
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori

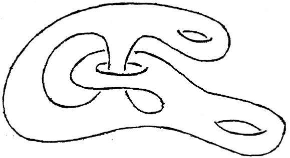
The Classification Theory has 4 Levels of Structure

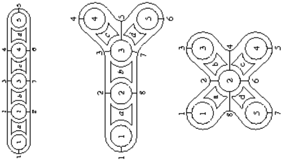
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
4. Extrinsic Embeddings

Four Levels of Structure









Poetic Organization

LINKS OF PERIODIC ORBITS

organize

BOUNDING TORI

organize

BRANCHED MANIFOLDS

organize

LINKS OF PERIODIC ORBITS