A Chaotic Walk with Friends

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Birthday Party
CORIA, France

June 23, 2011
Thank You to All My Friends

My colleagues and my friends — my colleagues are my friends — introduced me to and then helped to guide me through this new and delightful field.

Many are assembled here today.

To all I express my thanks for helping to make this such a festive occasion.
Outline

1. Background
2. Experimental Challenge
3. Topology of Orbits
4. Classification Challenge
5. Topological Analysis Program
6. Basis Sets of Orbits
7. Covers and Images
8. Bounding Tori
9. Representation Theory of Strange Attractors
10. Summary
Summary

Things to Come
Where it all began:
Hey Bob, these guys down the hall are doing fantastic stuff ...
Usual Culprits: 2

Where it all began:
My friends tell me you like to speak to experimentalists ...

Jorge Tredicce
“I like to ski the walls ....”
J. R. Tredicce

Can you explain my data?

I bet you can’t explain my data!
Motivation

Where is Tredicce coming from?

Feigenbaum:

\[ \alpha = 4.66920 16091 \ldots \]
\[ \delta = -2.50290 78750 \ldots \]
Laser with Modulated Losses
Experimental Arrangement
Ask the Masters: 1

What to Grab Hold of ??

Search for Invariants
Strange Attractor

How to Characterize a Strange Attractor

The $\Omega$ limit set of the flow. There are unstable periodic orbits “in” the strange attractor. They are

- “Abundant”
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor
Ask the Masters: 2 & 3

Periodic Orbits are the Key

Joseph Fourier
Linear Systems

Henri Poincare
Nonlinear Systems
UPOs: Skeletons of Strange Attractors

Here is a Strange Attractor (Belousov-Zhabotinskii Reaction)
Usual Culprits: 3 & 4

The Topological Team

Hernan G. Solari
(disguised as Cristal)

Gabriel B. Mindlin

A Chaotic Walk with Friends
Robert Gilmore

Introduction-01
Introduction-02
Introduction-03
Deep Background-01
Deep Background-02
Deep Background-03
Deep Background-04
Experimental
Here is a period-one orbit in the attractor.
UPOs: Skeletons of Strange Attractors

Period-1 and period-2 orbits from the attractor.
UPOs: Skeletons of Strange Attractors

Lots of them.
Quantitative Measures for Periodic Orbits ??

Carl Friedrich Gauss
They Link: Pairwise, 3-Wise, ...
Organization of UPOs in $R^3$:

**Gauss Linking Number**

$$LN(A, B) = \frac{1}{4\pi} \oint \oint \frac{(r_A - r_B) \cdot dr_A \times dr_B}{|r_A - r_B|^3}$$

# Interpretations of LN $\sim$ # Mathematicians in World
Linking Number of Two UPOs

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese
Compute Table of Experimental LN

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<tr>
<th>Orbit</th>
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*All indices are negative.
Mechanisms for Generating Chaos

Stretching and Folding

- Stretching
- Folding

Diagram showing the process of stretching and folding with labeled parts:
  - (a) squeeze
  - (b) stretch
  - (c) boundary layer
Mechanisms for Generating Chaos

Tearing and Squeezing
Systematics of Linking Number Tables

Joan S. Birman
Robert F. Williams

Construct a Branched Manifold
Collapse Along the Stable Manifold

Birman - Williams Projection

Identify \( x \) and \( y \) if

\[
\lim_{t \to \infty} |x(t) - y(t)| \to 0
\]
Ask the Masters: 7 & 8

Two Standard Strange Attractors

Otto Rössler

Edward Lorenz
A Very Common Mechanism

Rössler:

Attractor  Branched Manifold
A Mechanism with Symmetry

Lorenz: Attractor Branched Manifold
Examples of Branched Manifolds

Inequivalent Branched Manifolds

(a) 

(b) 

(c) 

(d)
Motion of Blobs in Phase Space

Stretching — Squeezing
Any branched manifold can be built up from stretching and squeezing units

subject to the conditions:

- Outputs to Inputs
- No Free Ends
Rössler System

(a) Rössler Equations

\[
\begin{align*}
\frac{dx}{dt} &= -y - z \\
\frac{dy}{dt} &= x + ay \\
\frac{dz}{dt} &= \beta + z(x - c)
\end{align*}
\]

(b) Graphs of x, y, and z over time.

(c) 3D trajectory of the Rössler system.
Lorenz System

(a) Lorenz Equations
\[
\begin{align*}
\frac{dx}{dt} &= -\sigma x + \sigma y \\
\frac{dy}{dt} &= R x - y - xz \\
\frac{dz}{dt} &= -bz + xy
\end{align*}
\]

(b) Time Series

(c) Phase Portrait
Dynamics and Topology

Poincaré Smiles at Us in $R^3$

- Determine organization of UPOs ⇒
- Determine branched manifold ⇒
- Determine equivalence class of $SA$
Topological Analysis Program

- Locate Periodic Orbits
- Create an Embedding
- Determine Topological Invariants (LN)
- Identify a Branched Manifold
- Verify the Branched Manifold

Additional Steps

- Model the Dynamics
- Validate the Model
More Team Members

Nicholas B. Tufillaro  
Mario A. Natiello
Locate UPOs

Method of Close Returns
Embeddings

\textbf{Embeddings (= Black Magic)}

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological\(^{\dagger}\)

None Good

We Demand a 3 Dimensional Embedding
Locate UPOs

An Embedding and Periodic Orbits

Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser; Right: Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese
The Topology of Chaos

Marc Lefranc
Compute Table of Experimental LN

Table 7.2  Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinsky data

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*aAll indices are negative.*
A Chaotic Walk with Friends

Robert Gilmore

Introduction

Deep Background

Experimental

Lefranc - Cargese

Compare w. LN From Various BM

**Table 2.1** Linking numbers for orbits to period five in Smale horseshoe dynamics.

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Compare Topological Invariants
Propose Branched Manifold

Guess Branched Manifold

Figure 7. “Combing” the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

Lefranc - Cargese
Comparison Step

Identification & ‘Confirmation’

- $BM$ Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion (Fail to Reject $H_0$)
Evolution Under Parameter Change
Perestroikas of Strange Attractors

Evolution Under Parameter Change

Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].
**Figure 16.** Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].
A Prediction Realized

**TABLE 1** – Folding processes characteristic of the different species of templates treated in this work

<table>
<thead>
<tr>
<th>Species</th>
<th>Horseshoe</th>
<th>Reverse horseshoe</th>
<th>Out-to-in spiral</th>
<th>In-to-out spiral</th>
<th>Staple</th>
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**Modulation frequency normalized to the natural frequency**

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<th>1/4</th>
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Used and Martin (2010)
Orbits Can be “Pruned”

There Are Some Missing Orbits

Lorenz

Shimizu-Morioka
There Are Some Missing Orbits

Francisco Papoff

Arimondo et al.

• The branched manifold remains unchanged,
• the spectrum of orbits on it changes.
An Ongoing Problem

Forcing Diagram - Horseshoe

Forcing of Horseshoe Orbits to Period 3

u - SEQUENCE ORDER

(a) Forcing Diagram - Horseshoe

(b) Forcing of Horseshoe Orbits to Period 3

PD

QOD: f
An Ongoing Problem

Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required
Ask the Masters: 9

What is the relation between symmetry groups & strange attractors?

Elie Cartan
Universal Covering Group

Cartan’s Theorem for Lie Groups

Simply connected Lie group \( \overline{G} \)

Multiply connected Lie groups

Linearization "LOG" (unique)

\( \overline{G}/D_1 \)

\( \overline{G}/D_2 \)

\( \cdots \)

\( \overline{G}/D_r \)

EXF (unique)

Lie algebra \( g \)
The Symmetry of Chaos

Christophe Letellier
Modding Out a Rotation Symmetry

\[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}
\rightarrow
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
=
\begin{pmatrix}
\text{Re} \ (X + iY)^2 \\
\text{Im} \ (X + iY)^2 \\
Z
\end{pmatrix}
\]
Creating a Cover with Symmetry

\[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}
\leftarrow
\begin{pmatrix}
\mathbf{u} \\
\mathbf{v} \\
\mathbf{w}
\end{pmatrix} = \begin{pmatrix}
\text{Re} \left( X + iY \right)^2 \\
\text{Im} \left( X + iY \right)^2 \\
Z
\end{pmatrix}
\]
Perestroikas of Branched Manifolds

Constraints on Branched Manifolds

- “Inflate” a strange attractor
- Union of $\epsilon$ ball around each point
- Boundary is surface of bounded 3D manifold
- Torus that bounds strange attractor
Ask the Masters: 10

How do we characterize surfaces?

Leonard Euler
Count holes: $\chi(\partial M)$
Torus and Genus

Torus, Longitudes, Meridians
What kind of “dressed tori” enclose strange attractors?

Tsvetelin D. Tsankov
Markov Matrices and Symmetric Cycles
Motivation

Some Genus-9 Bounding Tori
Labeling Bounding Tori

Poincaré section is disjoint union of g-1 disks

Transition matrix sum of two g-1 \times g-1 matrices

One is cyclic g-1 \times g-1 matrix

Other represents union of cycles

Labeling via (permutation) group theory
Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units

- Outputs to Inputs
- No Free Ends
- Colorless
Aufbau Princip for Bounding Tori

Application: Lorenz Dynamics, g=3
Construction of Poincaré Section

\[ \text{P. S.} = \text{Union} \]
\[ \# \text{ Components} = g-1 \]
The Growth is Exponential

TABLE I: Number of canonical bounding tori as a function of genus, \( g \).

<table>
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<tr>
<th>( g )</th>
<th>( N(g) )</th>
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\[ \text{Exponential Growth} \]

\[ \text{The Growth is Exponential} \]
Usual Culprits: 11

How quickly does the number of bounding tori increase with $g$?

Jacob Katriel
Magician with permutation group cycles.
The Growth is Exponential
The Entropy is log 3
Constraints Provided by Bounding Tori

Two possible branched manifolds in the torus with $g=4$. 

(a) 

(b) 

(c)
Extrinsic Embedding of Bounding Tori

Partial classification by links of homotopy group generators. Nightmare Numbers are Expected.
Could there be representation theory for strange attractors?

Of course! There is a representation theory for everything!
There is a representation theory for strange attractors!

What is it? How does it work?

Daniel J. Cross
Hard at work (pretending)!

Daniel J. Cross
Instructing us.
Usual Culprits: 13 & 14

After fixed points — Organizing curves?

Tim Jones
What?

Jean-Marc Ginoux
How?
Summary

1 Question Answered ⇒ 2 Questions Raised

We must be on the right track!
Our Hope

Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information from experimental signals.
There is now a classification theory for low-dimensional strange attractors.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to $\mathbb{R}^3$ only — for now
The Classification Theory has 4 Levels of Structure
The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
Four Levels of Structure

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
4. Extrinsic Embeddings
Four Levels of Structure
Poetic Organization

LINKS OF PERIODIC ORBITS
organize
BOUNDING TORI
organize
BRANCHED MANIFOLDS
organize
LINKS OF PERIODIC ORBITS