

# A Chaotic Walk with Friends

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Birthday Party  
CORIA, France

June 23, 2011

# Thank You

A Chaotic  
Walk with  
Friends

Robert  
Gilmore

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01

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Experimental

## Thank You to All My Friends

My colleagues and my friends — my colleagues *are* my friends — introduced me to and then helped to guide me through this new and delightful field.

Many are assembled here today.

To all I express my thanks for helping to make this such a festive occasion.

## Outline

- 1 Background
- 2 Experimental Challenge
- 3 Topology of Orbits
- 4 Classification Challenge
- 5 Topological Analysis Program
- 6 Basis Sets of Orbits
- 7 Covers and Images
- 8 Bounding Tori
- 9 Representation Theory of Strange Attractors
- 10 Summary

# Summary

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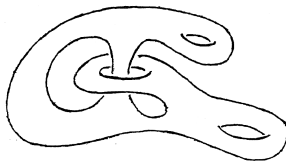
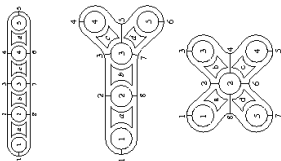
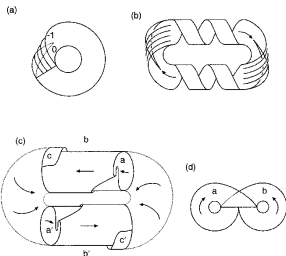
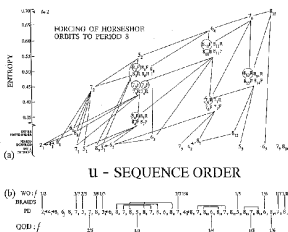
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## Things to Come





# Usual Culprits: 1

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## Where it all began:

Hey Bob, these guys down the hall are doing fantastic stuff ...



Elia Eschenazi  
The go-between

# Usual Culprits: 2

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## Where it all began:

My friends tell me you like to speak to experimentalists ...



Jorge Tredicce

“I like to ski the walls ....”

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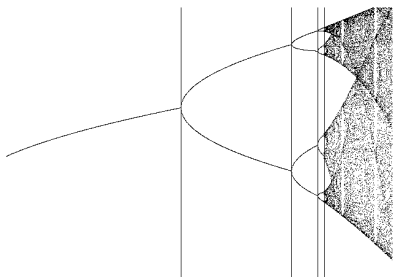
Experimental

## J. R. Tredicce

### Can you explain my data?

### I bet you can't explain my data!

## Where is Tredicce coming from?



**Feigenbaum:**

$$\alpha = 4.66920 16091 \dots$$

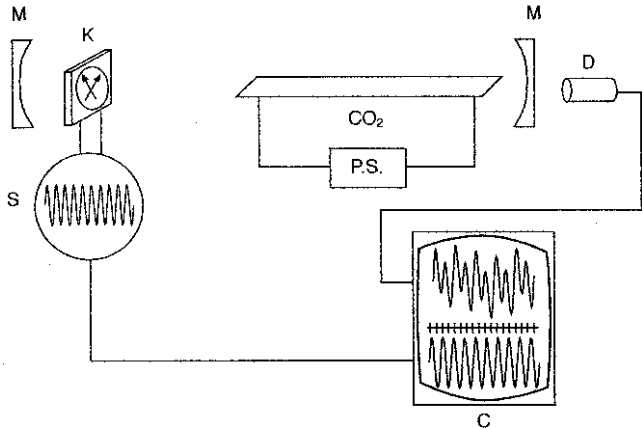
$$\delta = -2.50290 78750 \dots$$

# The Experiment

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## Laser with Modulated Losses Experimental Arrangement



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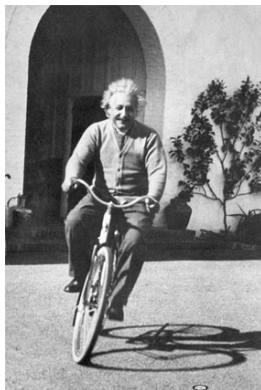
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## What to Grab Hold of ??



## Search for Invariants

# Strange Attractor

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## How to Characterize a Strange Attractor

The  $\Omega$  limit set of the flow. There are unstable periodic orbits “in” the strange attractor. They are

- “Abundant”
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor

## Periodic Orbits are the Key



Joseph Fourier  
Linear Systems



Henri Poincaré  
Nonlinear Systems

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# UPOs: Skeletons of Strange Attractors

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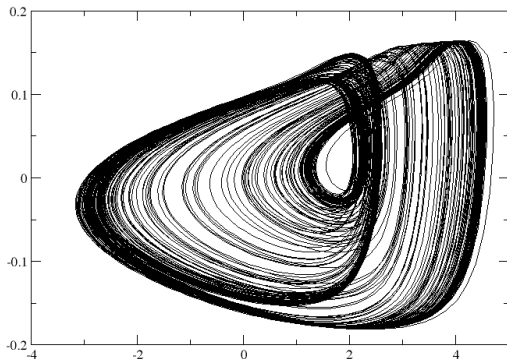
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Here is a Strange Attractor (Belousov-Zhabotinskii Reaction)

# Usual Culprits: 3 & 4

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## The Topological Team



Hernan G. Solari  
(disguised as Cristal)



Gabriel B. Mindlin

# UPOs: Skeletons of Strange Attractors

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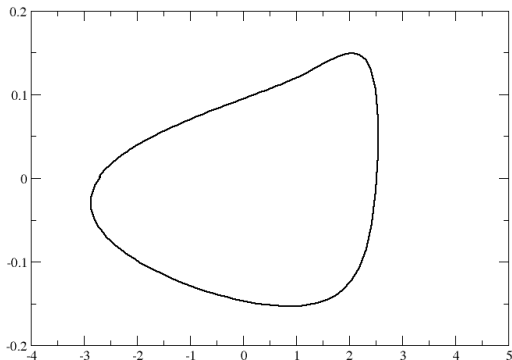
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Here is a period-one orbit in the attractor.

# UPOs: Skeletons of Strange Attractors

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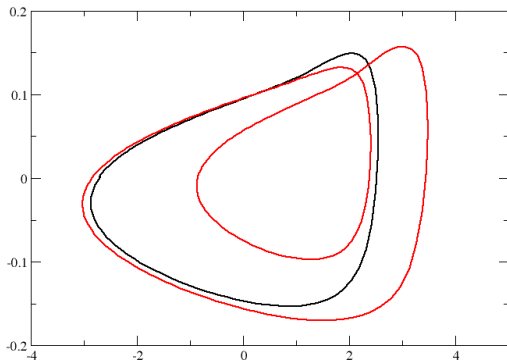
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Period-1 and period-2 orbits from the attractor.

# UPOs: Skeletons of Strange Attractors

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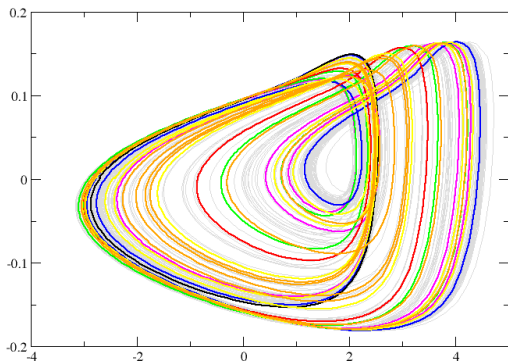
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Lots of them.

# Ask the Masters: 4

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Experimental

## Quantitative Measures for Periodic Orbits ??



Carl Friedrich Gauss

They Link: Pairwise, 3-Wise, ...

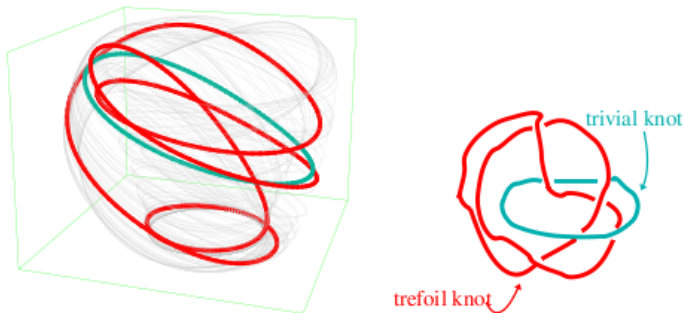
Organization of UPOs in  $R^3$ :

## Gauss Linking Number

$$LN(A, B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

# Interpretations of  $LN \simeq \#$  Mathematicians in World

## Linking Number of Two UPOs



**Figure 6.** Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese



# Determine Topological Invariants

## Compute Table of Experimental LN

**Table 7.2** Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data<sup>a</sup>

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

<sup>a</sup>All indices are negative.

# Mechanisms for Generating Chaos

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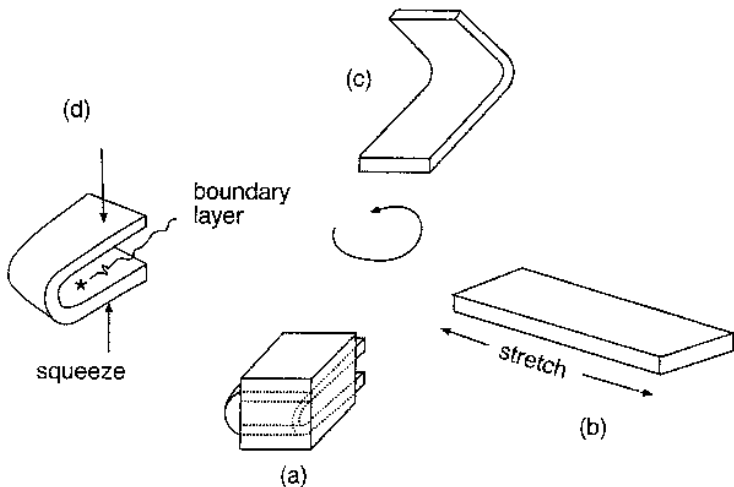
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## Stretching and Folding



# Mechanisms for Generating Chaos

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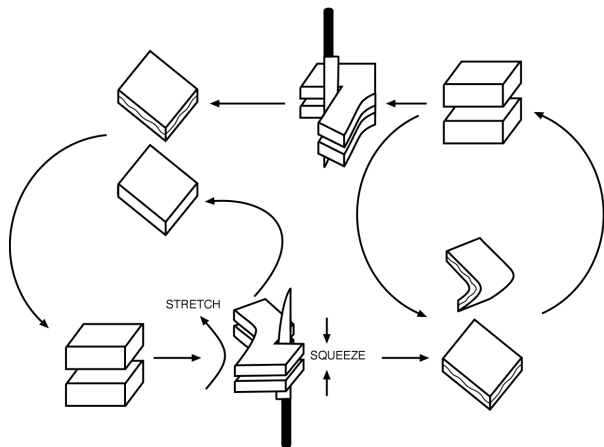
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## Tearing and Squeezing



# Ask the Masters: 5 & 6

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Experimental

## Systematics of Linking Number Tables



Joan S. Birman



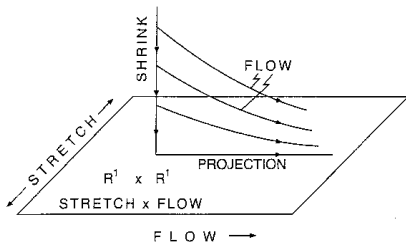
Robert F. Williams

## Construct a Branched Manifold

## Birman - Williams Projection

Identify  $x$  and  $y$  if

$$\lim_{t \rightarrow \infty} |x(t) - y(t)| \rightarrow 0$$



## Two Standard Strange Attractors



Otto Rössler



Edward Lorenz

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# A Very Common Mechanism

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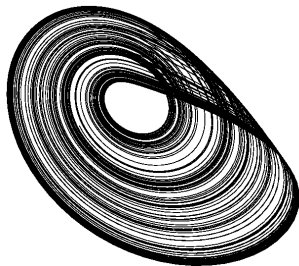
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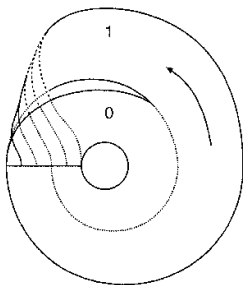
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## Rössler:

### Attractor



### Branched Manifold



# A Mechanism with Symmetry

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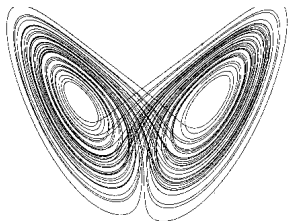
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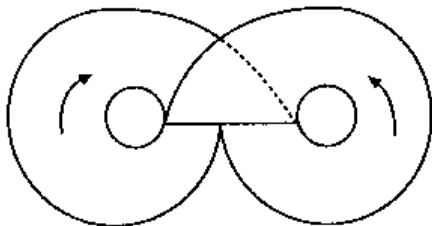
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## Lorenz:

### Attractor



### Branched Manifold

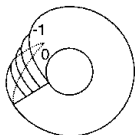




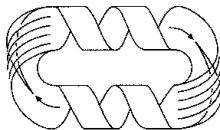
# Examples of Branched Manifolds

## Inequivalent Branched Manifolds

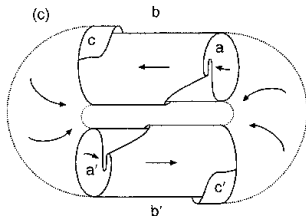
(a)



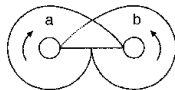
(b)



(c)



(d)

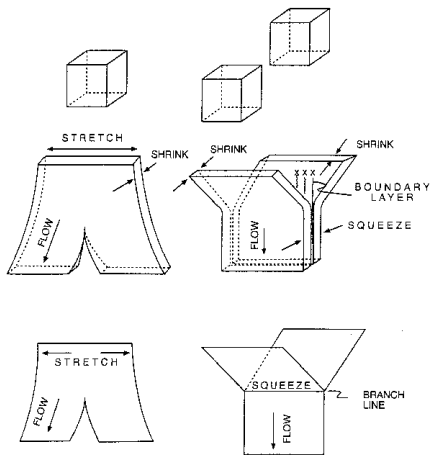


# Motion of Blobs in Phase Space

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## Stretching — Squeezing



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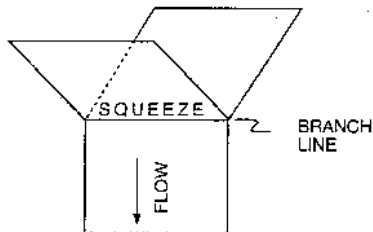
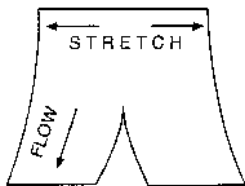
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# Aufbau Princip for Branched Manifolds

Any branched manifold can be built up from stretching and squeezing units



subject to the conditions:

- Outputs to Inputs
- No Free Ends

## Rössler System

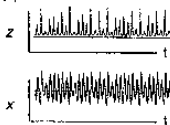
(a) Rössler Equations

$$\frac{dx}{dt} = -y - xz$$

$$\frac{dy}{dt} = x + ay$$

$$\frac{dz}{dt} = b + z(s - c)$$

(b)



(c)



(f)

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & +1 \end{pmatrix}$$

(e)



(f)



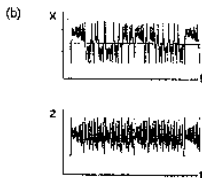
## Lorenz System

(a) Lorenz Equations

$$\frac{dx}{dt} = -\alpha x + \alpha y$$

$$\frac{dy}{dt} = \beta x - y - xz$$

$$\frac{dz}{dt} = -bz + xy$$

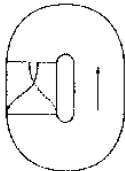


(f)

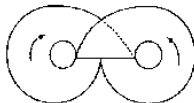
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

(e)



(d)



## Poincaré Smiles at $U$ s in $R^3$

- **Determine organization of UPOs  $\Rightarrow$**
- **Determine branched manifold  $\Rightarrow$**
- **Determine equivalence class of  $\mathcal{SA}$**

# We Like to be Organized

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## PERIODIC TABLE OF THE ELEMENTS

<http://www.kf-split.hr/periodic/en/>

PERIOD	GROUP																18	
	1 IA												17 VIIA		18 VIIIA			
1	H														He			
2	Li	Be											B	C	N	O	F	Ne
3	Na	Mg											Al	Si	P	S	Cl	Ar
4	K	Ca	Sc	Ti	V	Cr	Mn	Fe	Co	Ni	Cu	Zn	Ga	Ge	As	Se	Br	Kr
5	Rb	Sr	Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd	Ag	Cd	In	Sn	Sb	Te	I	Xe
6	Cs	Ba	La-Lu Lanthanide	Hf	Ta	W	Re	Os	Ir	Pt	Au	Hg	Tl	Pb	Bi	Po	At	Rn
7	Fr	Ra	Ac-Lr Actinide	Rf	Db	Sg	Bh	Hs	Mt	Uu	Uu	Uu	Uu	Uu	Uu	Uu	Uu	Uu
<p>RELATIVE ATOMIC MASS(1)</p> <p>GROUP IUPAC</p> <p>GROUP CAS</p> <p>ATOMIC NUMBER</p> <p>SYMBOL</p> <p>ELEMENT NAME</p> <p>Legend:</p> <ul style="list-style-type: none"> <li>Metal</li> <li>Semimetal</li> <li>Nonmetal</li> <li>Alkali metal</li> <li>Alkaline earth metal</li> <li>Transition metals</li> <li>Lanthanide</li> <li>Actinide</li> <li>Chalcogens element</li> <li>Halogens element</li> <li>Noble gas</li> </ul> <p>STANDARD STATE (100 °C; 101 kPa)</p> <ul style="list-style-type: none"> <li>Ne - gm</li> <li>Fe - solid</li> <li>Ga - liquid</li> <li>Ts - synthetic</li> </ul>																		

(1) Pure Appl. Chem., 73, No. 4, 987-993 (2001)

Relative atomic mass is shown with five significant figures. For elements with no stable nuclides, the value enclosed in brackets indicates the mass number of the longest-lived isotope of the element.

However three such elements (Tl, Po, and U) do have a characteristic terrestrial isotopic composition, and for these an atomic weight is tabulated.

Editor: Aditya Vardhan (adva@rediffmail.com)

Copyright © 1989-2003 ENIG (www.kf-split.hr)

### LANTHANIDE

57	138.91	58	140.12	59	140.91	60	144.24	61	(145)	62	150.36	63	151.96	64	157.25	65	158.93	66	162.50	67	164.93	68	167.26	69	168.93	70	173.04	71	174.97
La		Ce		Pr		Nd		Pm		Sm		Eu		Gd		Tb		Dy		Ho		Er		Tm		Yb		Lu	
LANTHANUM CERIUM PRASEODYMIUM NEODYMIUM PROMETHIUM SAMARIUM EUROPIUM GADOLINIUM TERBIUM DYSPROSIUM HOIMUM ERBIUM THULIUM YTTERIUM LUTETIUM																													

### ACTINIDE

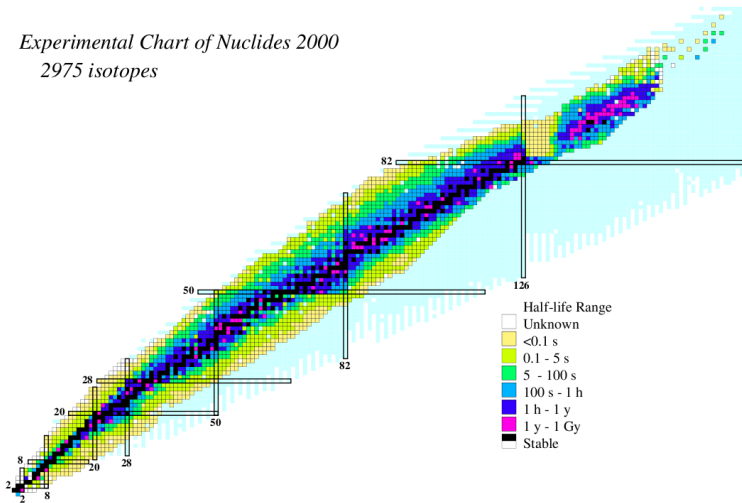
89	(227)	90	232.04	91	231.04	92	238.03	93	(237)	94	(244)	95	(243)	96	(247)	97	(247)	98	(251)	99	(252)	100	(257)	101	(258)	102	(259)	103	(262)
Ac		Th		Pa		U		Np		Pu		Am		Cm		Bk		Cf		Es		Fm		Md		No		Lr	
ACTINIUM THORIUM PROTACTINIUM URANIUM NEPTUNIUM PLUTONIUM AMERICIUM CURIUM BERKELIUM CALIFORNIUM ENSTENIUM FERMIUM MENDELEVIUM NOBELIUM LAWRENCIUM																													

# We Like to be Organized

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*Experimental Chart of Nuclides 2000*  
2975 isotopes



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## Topological Analysis Program

- Locate Periodic Orbits
- Create an Embedding
- Determine Topological Invariants (LN)
- Identify a Branched Manifold
- Verify the Branched Manifold

## Additional Steps

- Model the Dynamics
- Validate the Model

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# Usual Culprits: 5 & 6

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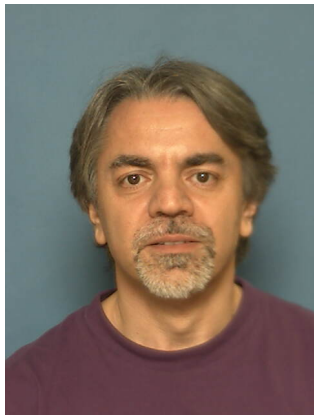
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Experimental

## More Team Members

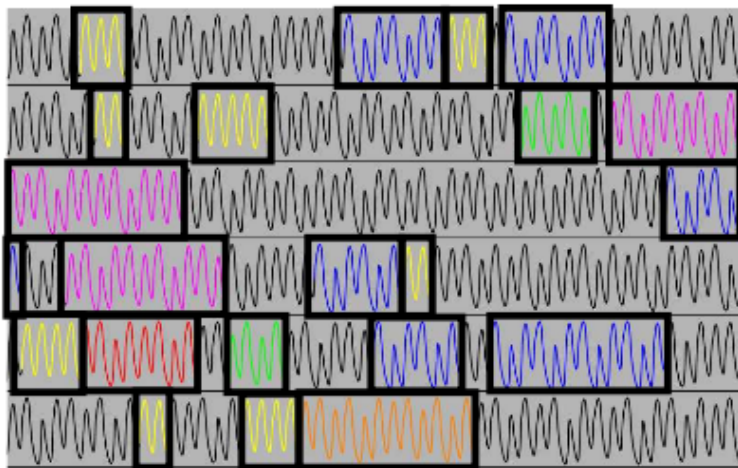


Nicholas B. Tufillaro



Mario A. Natiello

## Method of Close Returns



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## Embeddings (= Black Magic)

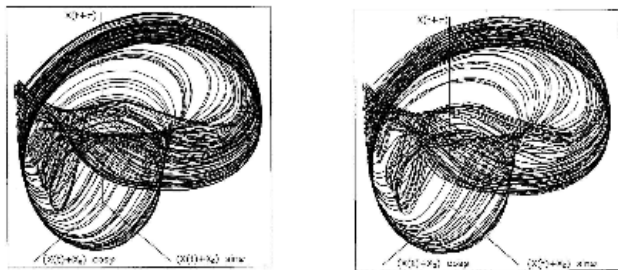
Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological<sup>†</sup>

None Good

We Demand a 3 Dimensional Embedding

## An Embedding and Periodic Orbits



**Figure 5.** Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese

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## The Topology of Chaos



Marc Lefranc

# Tabulate Topological Invariants

## Compute Table of Experimental LN

**Table 7.2** Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data<sup>a</sup>

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

<sup>a</sup>All indices are negative.

# Compare Topological Invariants

## Compare w. LN From Various $BM$

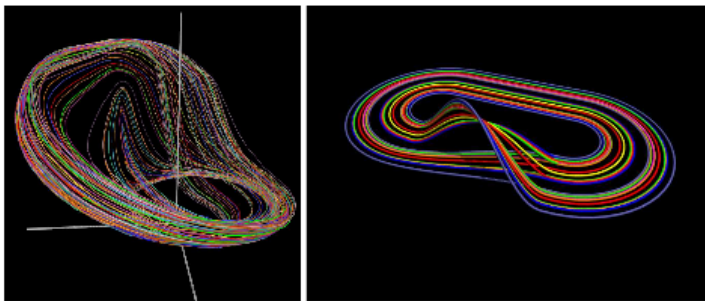
**Table 2.1** Linking numbers for orbits to period five in Smale horseshoe dynamics.

	$1^s$	$1^f$	$2_1$	$3^f$	$3^s$	$4_1$	$4_2^f$	$4_2^s$	$5_2^f$	$5_2^s$	$5_2^f$	$5_2^s$	$5_1^f$	$5_1^s$
Introduction-01	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Introduction-02	1	0	0	1	1	1	2	1	1	1	1	2	2	2
Introduction-03	01	0	1	1	2	2	3	2	2	2	2	3	3	4
Deep Background-01	001	0	1	2	2	3	4	3	3	3	3	4	4	5
Deep Background-02	011	0	1	2	3	2	4	3	3	3	3	5	5	5
Deep Background-03	0111	0	2	3	4	4	5	4	4	4	4	7	7	8
Deep Background-04	0001	0	1	2	3	3	4	3	4	4	4	5	5	5
Deep Background-05	0011	0	1	2	3	3	4	4	3	4	4	5	5	5
Deep Background-06	00001	0	1	2	3	3	4	4	4	4	5	5	5	5
Deep Background-07	00011	0	1	2	3	3	4	4	4	5	4	5	5	5
Deep Background-08	00111	0	2	3	4	5	7	5	5	5	5	6	7	8
Deep Background-09	00101	0	2	3	4	5	7	5	5	5	5	7	6	8
Deep Background-10	01101	0	2	4	5	5	8	5	5	5	5	8	8	8
Deep Background-11	01111	0	2	4	5	5	8	5	5	5	5	9	9	10



# Propose Branched Manifold

## Guess Branched Manifold



**Figure 7.** “Combing” the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

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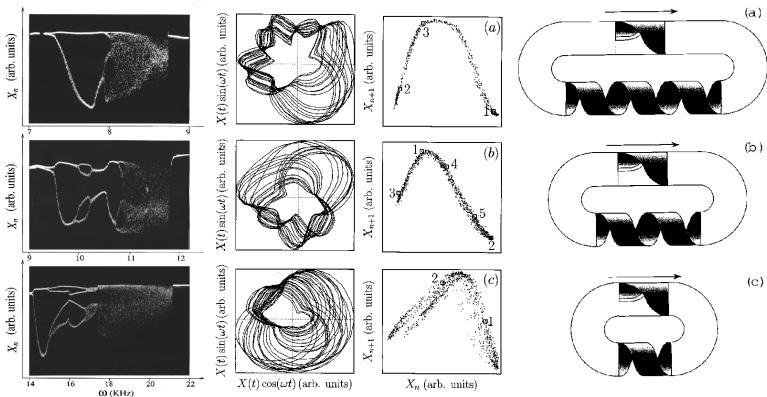
Experimental

## Identification & ‘Confirmation’

- $\mathcal{BM}$  Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion (Fail to Reject  $H_0$ )

# A Perestroika

## Evolution Under Parameter Change



# Perestroikas of Strange Attractors

## Evolution Under Parameter Change

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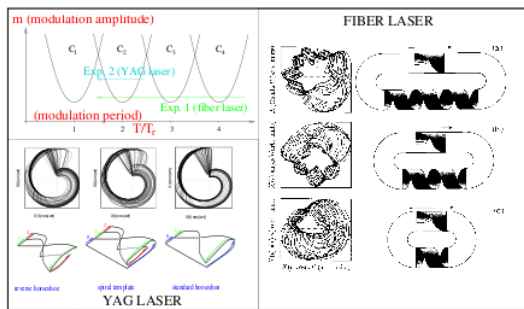
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Experimental



**Figure 11.** Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

# An Unexpected Benefit

## Analysis of Nonstationary Data

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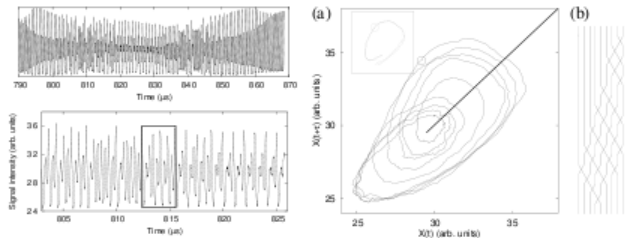
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Experimental



**Figure 16.** Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is  $h_T = 0.377$ , showing that the underlying dynamics is chaotic. Reprinted from [61].

Lefranc - Cargese

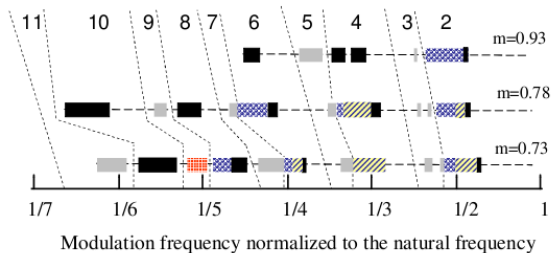
# A Prediction Realized

A Chaotic Walk with Friends

Robert Gilmore

TABLE 1 – Folding processes characteristic of the different species of templates treated in this work

Species	Horseshoe	Reverse horseshoe	Out-to-in spiral	In-to-out spiral	Staple	S
Code in Fig 1				Not found here		
Insertion matrix	(0 1)	(1 0)	(0 2 1)	(1 2 0)	(0 2 1) or (1 2 0)	(2 1 0)
Sketch of the folding process						



Used and Martin (2010)

Introduction-01

Introduction-02

Introduction-03

Deep Background-01

Deep Background-02

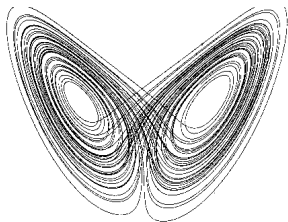
Deep Background-03

Deep Background-04

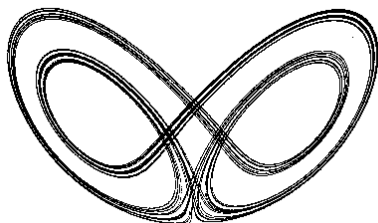
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# Orbits Can be “Pruned”

## There Are Some Missing Orbits



Lorenz



Shimizu-Morioka

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# Usual Culprits: 8

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Experimental

## There Are Some Missing Orbits



Francisco Papoff

Arimondo et al.

- The branched manifold remains unchanged,
- the spectrum of orbits on it changes.



# An Ongoing Problem

## Forcing Diagram - Horseshoe

A Chaotic Walk with Friends

Robert Gilmore

Introduction-01

Introduction-02

Introduction-03

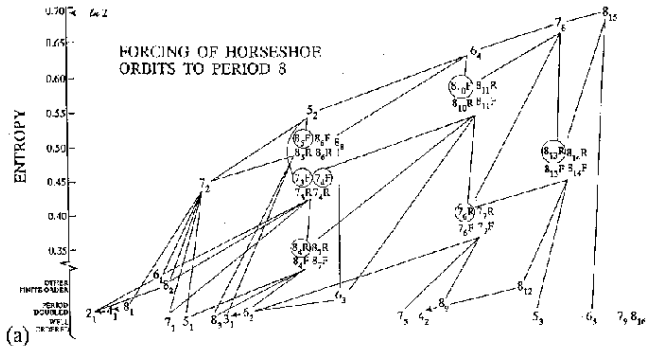
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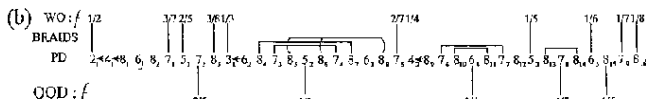
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Experimental



### U - SEQUENCE ORDER



## Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required

# Ask the Masters: 9

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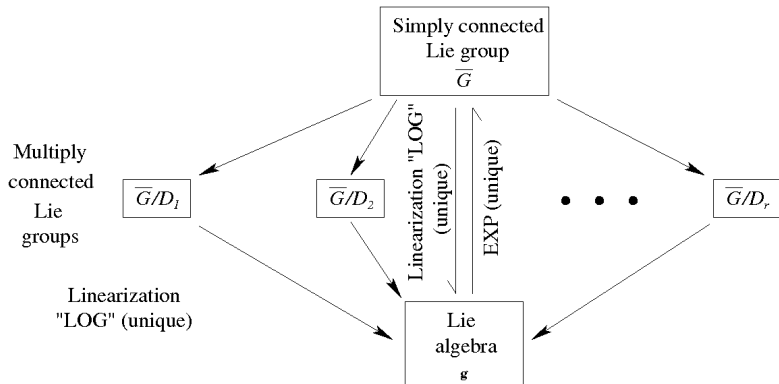
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What is the relation between  
symmetry groups & strange attractors?



Elie Cartan

## Cartan's Theorem for Lie Groups



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## The Symmetry of Chaos

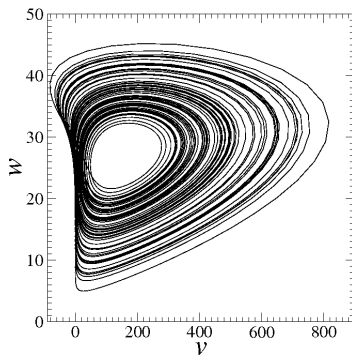
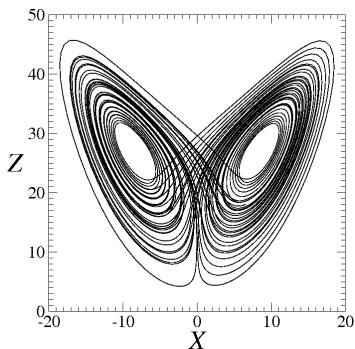


Christophe Letellier

# Modding Out a Rotation Symmetry

## Modding Out a Rotation Symmetry

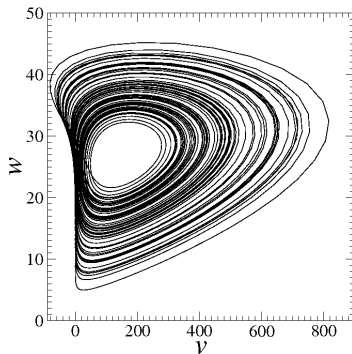
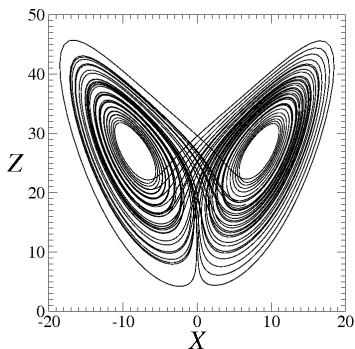
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \operatorname{Re} (X + iY)^2 \\ \operatorname{Im} (X + iY)^2 \\ Z \end{pmatrix}$$



# Lifting an Attractor: Cover-Image Relations

## Creating a Cover with Symmetry

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \leftarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \operatorname{Re} (X + iY)^2 \\ \operatorname{Im} (X + iY)^2 \\ Z \end{pmatrix}$$



## Cover-Image Branched Manifolds

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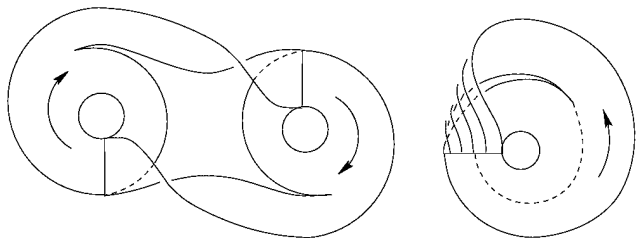
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Experimental





## Constraints on Branched Manifolds

- **“Inflate” a strange attractor**
- **Union of  $\epsilon$  ball around each point**
- **Boundary is surface of bounded 3D manifold**
- **Torus that bounds strange attractor**

# Ask the Masters: 10

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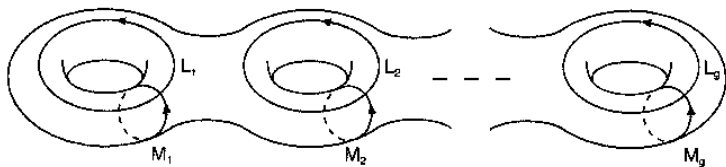
How do we characterize surfaces?



Leonard Euler  
Count holes:  $\chi(\partial\mathcal{M})$

# Torus and Genus

## Torus, Longitudes, Meridians



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# Usual Culprits: 10

What kind of “dressed tori” enclose strange attractors?



Tsvetelin D. Tsankov  
Markov Matrices and Symmetric Cycles

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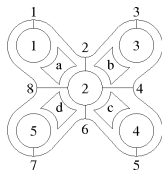
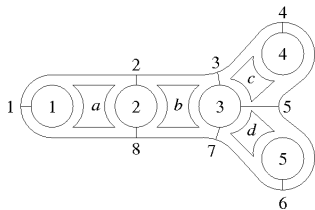
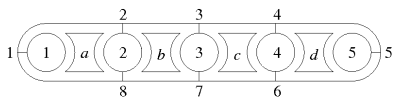
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Experimental

## Some Genus-9 Bounding Tori



## Labeling Bounding Tori

**Poincaré section is disjoint union of  $g-1$  disks**

**Transition matrix sum of two  $g-1 \times g-1$  matrices**

**One is cyclic  $g-1 \times g-1$  matrix**

**Other represents union of cycles**

**Labeling via (permutation) group theory**

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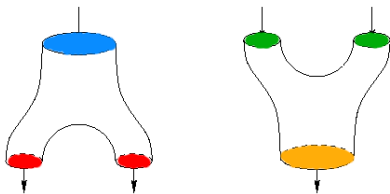
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Experimental

# Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units



- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

# Aufbau Princip for Bounding Tori

## Application: Lorenz Dynamics, $g=3$

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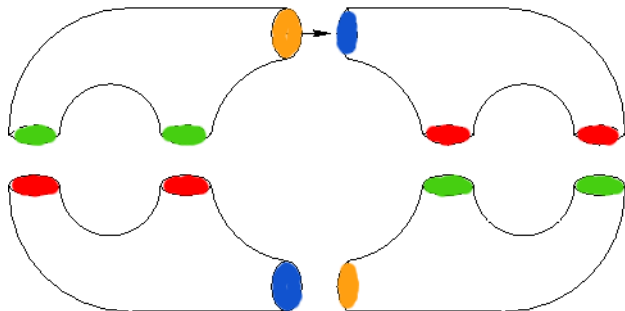
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## Construction of Poincaré Section

P. S. = Union 

# Components =  $g-1$

# Exponential Growth

## The Growth is Exponential

TABLE I: Number of canonical bounding tori as a function of genus,  $g$ .

$g$	$N(g)$	$g$	$N(g)$	$g$	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

TABLE I: Enumeration of canonical forms up to genus 9

$g$	$m$	$(p_1, p_2, \dots, p_m)$	$n_1 n_2 \dots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212221
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	1111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	1212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11311313
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

# Usual Culprits: 11

How quickly does the number of bounding tori increase with  $g$ ?



Jacob Katriel

Magician with permutation group cycles.

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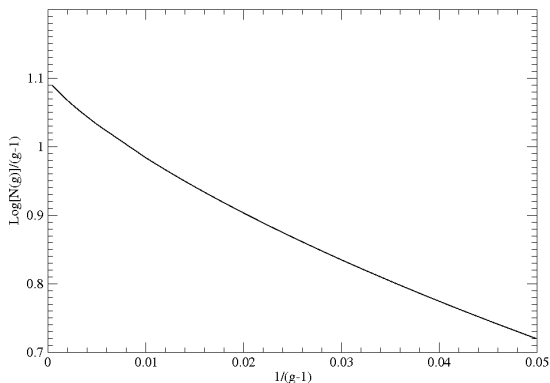
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## The Growth is Exponential The Entropy is $\log 3$

Bounding Torus Entropy

$\text{Log}[N(g)]/(g-1)$



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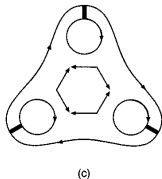
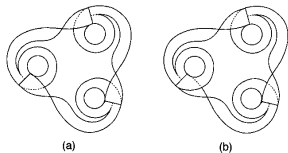
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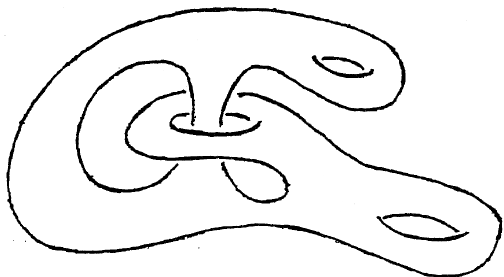
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Experimental

## Two possible branched manifolds in the torus with $g=4$ .



## Extrinsic Embedding of Intrinsic Tori



Partial classification by links of homotopy group generators.  
Nightmare Numbers are Expected.

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Experimental

# Ask the Masters: 11

Could there be representation theory for strange attractors?



Eugene Wigner

Of course! There is a representation theory for everything!

# Usual Culprits: 12

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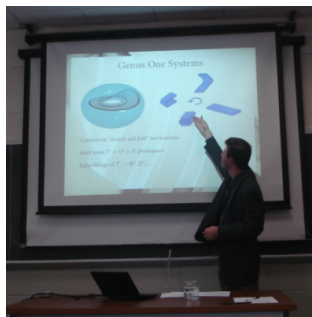
Experimental

There is a representation theory for strange attractors!

What is it? How does it work?



Daniel J. Cross  
Hard at work (pretending)!



Daniel J. Cross  
Instructing us.



# Usual Culprits: 13 & 14

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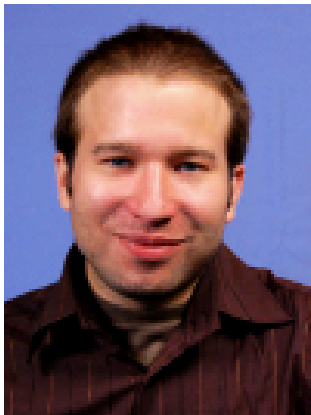
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Experimental

After fixed points — Organizing curves?



Tim Jones  
What?



Jean-Marc Ginoux  
How ?

## Summary

**1 Question Answered  $\Rightarrow$   
2 Questions Raised**

**We must be on the right track !**

# Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

# Result

**There is now a classification theory  
for low-dimensional strange attractors.**

- ① It is topological
- ② It has a hierarchy of 4 levels
- ③ Each is discrete
- ④ There is rigidity and degrees of freedom
- ⑤ It is applicable to  $R^3$  only — for now

# The Classification Theory has 4 Levels of Structure

A Chaotic  
Walk with  
Friends

Robert  
Gilmore

Introduction-  
01

Introduction-  
02

Introduction-  
03

Deep  
Background-  
01

Deep  
Background-  
02

Deep  
Background-  
03

Deep  
Background-  
04

Experimental

# The Classification Theory has 4 Levels of Structure

### ① Basis Sets of Orbits

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Experimental

# The Classification Theory has 4 Levels of Structure

- 1 Basis Sets of Orbits
- 2 Branched Manifolds

# The Classification Theory has 4 Levels of Structure

- 1 Basis Sets of Orbits
- 2 Branched Manifolds
- 3 Bounding Tori

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# The Classification Theory has 4 Levels of Structure

- 1 Basis Sets of Orbits
- 2 Branched Manifolds
- 3 Bounding Tori
- 4 Extrinsic Embeddings

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Experimental

# Four Levels of Structure

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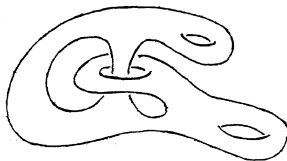
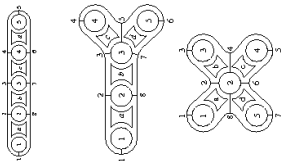
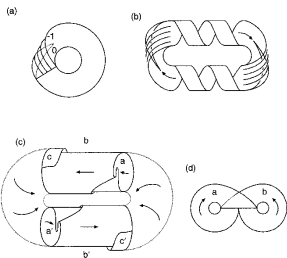
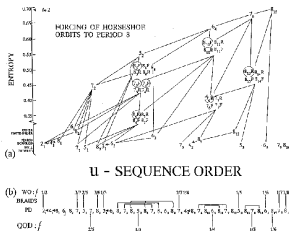
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Deep  
Background-  
04

Experimental



# Poetic Organization

**LINKS OF PERIODIC ORBITS**

**organize**

**BOUNDING TORI**

**organize**

**BRANCHED MANIFOLDS**

**organize**

**LINKS OF PERIODIC ORBITS**

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Experimental