

The Two-body Problem

The two-body problem—determining the motion of two bodies orbiting one another under their mutual gravitational attraction—is perhaps the best-known problem in gravitational dynamics. Here we show how it can be reduced to an equivalent one-body problem and then solved as a special case of motion in a potential field.

The equations of motion for two bodies of masses m_1 and m_2 , positions \mathbf{x}_1 and \mathbf{x}_2 , and velocities \mathbf{v}_1 and \mathbf{v}_2 moving under gravity are

$$m_1 \mathbf{a}_1 = \frac{Gm_1 m_2}{r_{12}^3} (\mathbf{x}_2 - \mathbf{x}_1) \quad (1)$$

$$m_2 \mathbf{a}_2 = \frac{Gm_1 m_2}{r_{12}^3} (\mathbf{x}_1 - \mathbf{x}_2), \quad (2)$$

where $\mathbf{a} = \ddot{\mathbf{x}}$ and $r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$. Adding these two equations, we see immediately that $m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 = 0$, so (integrating once) $m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = \text{constant}$ and the center of mass of the bodies moves at constant velocity. Dividing Equation (1) by m_1 , Equation (2) by m_2 , and subtracting, we obtain the equation of motion for the relative separation $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$:

$$\ddot{\mathbf{r}} = -\frac{GM\mathbf{r}}{r_{12}^3} \quad (3)$$

where $M = m_1 + m_2$. Thus the relative motion in the two-body problem is identical to motion of a test particle in the potential field of a mass M .

As we saw in class, motion in a central potential [$\phi(r)$ say] is planar, since angular momentum is conserved, and in polar coordinates (r, θ) in that plane, the equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = F(r) \quad (4)$$

$$r^2 \dot{\theta} = L = \text{constant}, \quad (5)$$

where $F(r) = -\phi'(r)$ and the second expression simply states conservation of angular momentum. We can use the angular momentum integral (5) to simplify the radial equation (4) by eliminating the transverse motion $\dot{\theta}$:

$$\ddot{r} = \frac{L^2}{r^3} + F(r) = -\frac{d\phi_{\text{eff}}}{dr}, \quad (6)$$

where the *effective potential* $\phi_{\text{eff}} = \phi + \frac{1}{2}L^2/r^2$ combines the external and the centrifugal forces into a single quantity.

We can also use Equation (5) to eliminate time as a variable in Equation (4) by writing

$$\frac{d}{dt} = \frac{L}{r^2} \frac{d}{d\theta} \quad (7)$$

so the radial equation becomes

$$\frac{L^2}{r^2} \frac{d}{d\theta} \left(\frac{1}{r^2} \frac{dr}{d\theta} \right) - \frac{L^2}{r^3} = F(r). \quad (8)$$

We now make the key substitution $u = 1/r$, so

$$\frac{dr}{d\theta} = -\frac{1}{u^2} \frac{du}{d\theta} = -r^2 \frac{du}{d\theta} \quad (9)$$

and the radial equation (8) becomes

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{L^2u^2}. \quad (10)$$

The first integral of this equation, obtained by multiplying by $du/d\theta$ and integrating with respect to θ (and noting that $d\phi/dr = u^2 d\phi/du$) is

$$\frac{1}{2} \left(\frac{du}{d\theta} \right)^2 + \frac{1}{2}u^2 + \frac{\phi}{L^2} = \text{constant}. \quad (11)$$

This is often called the *radial energy equation*. Since

$$\frac{du}{d\theta} = \frac{1}{Lu^2} \frac{du}{dt} = -\frac{1}{L} \frac{dr}{dt} = -\frac{v_r}{L}, \quad (12)$$

where v_r the radial velocity, and since the transverse velocity is $v_\theta = r\dot{\theta} = L/r = Lu$, it should be clear that the constant on the right side of Equation (11) is just E/L^2 , where

$$E = \frac{1}{2}v_r^2 + \frac{1}{2}v_\theta^2 + \phi(r) \quad (13)$$

is the energy per unit mass.

The equations become particularly simple for the so-called *Kepler problem* describing motion under gravity, since

$$F(r) = -\frac{GM}{r^2} = -GMu^2 \quad (14)$$

and Equation (8) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{L^2}. \quad (15)$$

The solution to Equation (15) is

$$u = C \cos(\theta - \theta_0) + \frac{GM}{L^2}, \quad (16)$$

where C and θ_0 are constants. Writing

$$e = \frac{CL^2}{GM} \quad (17)$$

$$a = \frac{L^2}{GM(1 - e^2)}, \quad (18)$$

Equation (16) becomes

$$r [1 + e \cos(\theta - \theta_0)] = a(1 - e^2), \quad (19)$$

which is the standard equation of an ellipse of semi-major axis a and eccentricity e , with the mass M at one focus and the long axis oriented along the direction $\theta = \theta_0$. For $e < 1$, r is bounded and the orbit is a closed ellipse. For $e \geq 1$ it is possible for $r \rightarrow \infty$ and the orbit is open—a hyperbola ($e > 1$) or a parabola ($e = 1$). By substituting Equation (16) into the radial energy equation (11) and using equations (17) and (18), it is easily verified that the total energy (per unit mass) of the motion is

$$E = -\frac{GM}{2a}. \quad (20)$$

The shape of the ellipse is completely specified by the geometric parameters a and e , or, equivalently, by the dynamical parameters E and L . Note that a hyperbolic solution with $e > 1$ corresponds to $a < 0$ and $E > 0$, so the dividing line between bound and unbound motion is $E = 0$. We can use Equation (20) to eliminate C from Equation (17) to find

$$e^2 = 1 + \frac{2EL^2}{G^2M^2}. \quad (21)$$

It is not possible to write down a closed-form analytic expression for r as a function of time t , but we can derive a parametric solution to the problem, as follows. We assume $E < 0$. A similar derivation holds for the unbound case. Writing

$$E = \frac{1}{2}v_r^2 + \frac{1}{2}v_\theta^2 - \frac{GM}{r} = \frac{1}{2}v_r^2 + \frac{1}{2}\frac{L^2}{r^2} - \frac{GM}{r}, \quad (22)$$

we can rearrange to find

$$v_r^2 = 2\left(E + \frac{GM}{r}\right) - \frac{L^2}{r^2} \quad (23)$$

$$= -\frac{2E}{r^2} \left(-r^2 - \frac{GM}{2E}r + \frac{L^2}{2E}\right). \quad (24)$$

But the solutions to this quadratic in r are known, since $v_r = 0$ at the two turning points of the orbit, corresponding to pericenter, $\theta = \theta_0, r = r_p = a(1 - e)$, and apocenter, $\theta = \theta_0 \pm \pi, r = r_a = a(1 + e)$. The quadratic must therefore factorize as $(r_a - r)(r - r_p)$, and we have

$$v_r^2 = -\frac{2E}{r^2} (r_a - r)(r - r_p). \quad (25)$$

Since $v_r = dr/dt$, we can say

$$t = \int \frac{dr}{v_r} \quad (26)$$

$$= \frac{1}{\sqrt{-2E}} \int \frac{r dr}{\sqrt{(r_a - r)(r - r_p)}}. \quad (27)$$

We can do the integral by using the substitution $r = a(1 - e \cos \eta)$. Then $dr = ae \sin \eta d\eta$ and $r_a - r = ae(1 + \cos \eta)$, $r - r_p = ae(1 - \cos \eta)$ so the denominator is $ae \sin \eta$. The result is

$$t = \frac{a}{\sqrt{-2E}} \int (1 - e \cos \eta) d\eta \quad (28)$$

$$= \frac{a}{\sqrt{-2E}} (\eta - e \sin \eta). \quad (29)$$

Note that the definition of η means that $v_r < 0$ for $\eta < 0$ and $v_r > 0$ for $\eta > 0$, so this result is correct over the entire orbit.

Writing $E = -GM/2a$, the leading coefficient is $\sqrt{a^3/GM} \equiv \Omega^{-1}$, where Ω is the *mean motion*. Thus we have a complete parametric solution

$$\Omega t = \eta - e \sin \eta \quad (30)$$

$$r = a(1 - e \cos \eta). \quad (31)$$

Equation (30) is called *Kepler's equation*. It must be solved numerically for η given t . The parameter η is called the *eccentric anomaly*. Once r is known from Equation (31), θ can be determined from Equation (19). Finally, we note that the period of the motion is

$$T = \frac{2\pi}{\Omega} = 2\pi \left(\frac{a^3}{GM} \right)^{1/2}, \quad (32)$$

which is Kepler's Third Law.