PHYS 431/531: Galactic Dynamics

Fall 2011, Homework #4

(Due November 29, 2011)

1. (a) A particle of mass m and speed v encounters a body of mass $M \gg m$ and speed V on an unbound trajectory. By considering the scattering event in the rest frame of the more massive body, show that the maximum possible change in the *speed* of the lighter body is $2V$. Under what circumstances is this maximum achieved?

   (b) The velocities of stars in a stellar system are described by a three-dimensional Maxwellian distribution—that is, the number of stars with speeds between v and $v + dv$ is $f(v)\, dv$, where

   $$f(v) \;=\; Av^2\, e^{-mv^2/2kT}.$$

   Here, A is a normalization constant, defined so that $\int_0^\infty f(v)dv = N$, m is the stellar mass, assumed constant, k is Boltzmann's constant, and T is the temperature of the system. Verify the statement made in class that the mean stellar kinetic energy is $\frac{1}{2}m\langle v^2\rangle \;=\; \frac{3}{2}kT$.

2. Work out the details of the simple evaporative model discussed in class. Stars evaporate from a cluster of mass M on a time scale $t_{ev} = \alpha t_R$, where $\alpha \gg 1$, so

   $$\frac{dM}{dt} = -\frac{M}{\alpha t_R}\,. \tag{1}$$

   For pure evaporation, each escaping star carries off exactly zero energy (i.e. stars barely escape the cluster potential), so the total energy of the cluster remains constant.

   (a) If the cluster potential energy can always be written as $U = -k\frac{GM^2}{2R}$ for fixed k, where R is a characteristic cluster radius, and assuming that the cluster is always in virial equilibrium, show that $R \propto M^2$ as the cluster evolves.

   (b) Assuming that the relaxation time t_R scales as $M^{1/2}R^{3/2}$, so

   $$t_R = t_{R0}\left(\frac{M}{M_0}\right)^{1/2}\left(\frac{R}{R_0}\right)^{3/2},$$

   solve equation (1) to determine the lifetime of the cluster (in terms of its initial relaxation time t_{R0}). Also write down an expression for the mean cluster density as a function of time.

   (c) Estimate this lifetime for a globular cluster of mass $5 \times 10^5 M_\odot$ and radius 10 pc.

3. A satellite galaxy of mass M_s moves in a circular orbit of radius R in a spherically symmetric galactic halo of density $\rho(r) = v_c^2/4\pi G r^2$, with $M_s \ll v_c^2 R/G$. The stars (and dark matter particles) in the parent galaxy all have masses much less than M_s.

   (a) Use the equation for dynamical friction given by Sparke & Gallagher (Eq. 7.8) to write down the drag force on the satellite as it orbits.

   (b) The satellite sinks inward so slowly that it can be thought of as moving through a series of circular orbits, so its orbital speed at any radius r is always equal to the circular orbital speed at r. What is the angular momentum $L(r)$ of the satellite at radius r?

(c) By equating the rate of change of L to the torque exerted on the satellite by dynamical friction, show that the distance $r(t)$ from the satellite to the center of the galaxy obeys the differential equation

$$\frac{dr}{dt} = -\frac{GM_s \ln \Lambda}{v_c\, r}.$$

(d) Solve this equation to estimate the time taken for the satellite to sink to the center of the parent galaxy.

(e) Evaluate this time for a hypothetical "Magellanic Cloud" with $M_s = 2 \times 10^{10} M_\odot$ on an initially circular orbit of radius $R = 50\,\mathrm{kpc}$ around our Galaxy, with $v_c = 220\,\mathrm{km/s}$. Take $\Lambda = 20$.

4. *[Undergraduates only]* Show that, if the distribution function in a spherically symmetric system is a function of energy only, then the velocity distribution is isotropic.

5. *[Graduate students only]* A stellar system in which all particles are on radial orbits is described by the distribution function

$$f(\mathcal{E}, L) = \begin{cases} A\delta(L^2)(\mathcal{E} - \mathcal{E}_0)^{-1/2} & (\mathcal{E} > \mathcal{E}_0) \\ 0 & (\mathcal{E} \leq \mathcal{E}_0), \end{cases}$$

where $\mathcal{E} = \psi - \frac{1}{2}v^2$ is relative energy and $\mathcal{E}_0$ and A are constants.

(a) By writing $v^2 = v_r^2 + v_t^2$, where v_r and v_t are the radial and transverse velocities, and $L = rv_t$, prove that the volume element d^3v may be written

$$d^3v = 2\pi v_t dv_t dv_r = \frac{\pi d\mathcal{E} dX}{r^2 v_r},$$

where $X = L^2$.

(b) Hence show that the density is

$$\rho(r) = \begin{cases} Br^{-2} & (r < r_0) \\ 0 & (r \geq r_0), \end{cases}$$

where B is a constant and the relative potential at r_0 satisfies $\psi(r_0) = \mathcal{E}_0$.