PHYS 501: Mathematical Physics I

Fall 2011

Homework #6

(Problems 1 and 2 due December 2, 2011, 5 pm)

(Problems 3 and 4 due December 8, 2011, 5 pm)

1. Use contour integration to find the inverse Fourier transform $f(t)$ of the function

$$F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\sin\omega a}{\omega}$$

(where $a > 0$), for all values of t. Recall that F was obtained as the Fourier transform of of a step function with a discontinuity at $|t| = a$. What is the value of $f(a)$?

2. Consider the solution to the ordinary differential equation

$$\frac{d^2y}{dx^2} + xy \;=\; 0$$

for which $|y| \to 0$ as $|x| \to \infty$. (This equation is known as the *Airy equation*. It appears in the theory of the diffraction of light.)

(a) Sketch this solution. Specifically, what behavior do you expect as $x \to -\infty$ and $x \to +\infty$?

(b) By Fourier transforming the above equation, determine $Y(\omega)$, the Fourier transform of $y(x)$, and hence write down an integral expression for $y(x)$. (Hint: What is the inverse transform of $Y'(\omega)$?)

3. (a) Let $\{\mathcal{R}_j\}$ be a random sequence of real numbers, with $\mathcal{R}_j$ distributed uniformly between -1 and 1. For the N-point discrete Fourier transform of $\{\mathcal{R}_j\}$:

$$r_k = \sum_{j=0}^{N-1} \mathcal{R}_j\, e^{2\pi ijk/N}\,,$$

calculate the expectation value and variance of the "periodogram estimate" of the (unnormalized) power spectrum, $P_k = |r_k|^2 + |r_{N-k}|^2$, for $k = 1, \ldots, N/2$.

(b) Generate a sequence of random numbers (using, for example, the `random` function in Linux, or `ran2` in Numerical Recipes in C, p. 282) with properties as in part (a), and compute $\{P_k\}$ numerically using a fast Fourier transform with $N = 32768$. Plot first P_k, then $\log_{10} P_k$ against $\log_{10} k$, for $k = 1, \ldots, N/2$.

How does your graph compare with analytic expectations? Repeat the calculation, averaging the plotted data over an interval of width 64 centered on each data point.

(c) Repeat the computation in part (b) for a *random walk* $\{w_j\}$ defined by $w_0 = 0$, $w_{j+1} = w_j + \mathcal{R}_j$. Can you account for the differences in appearance between this graph and one you obtained in part (b)?

4. A “corrupted” real-valued dataset may be found in the file `corrupt.dat` on the PHYS 501 Web site. It is a time sequence consisting of two columns of data, j and c_j, for $j = 0, \ldots, N-1$. The original data have been convolved with a (normalized) Gaussian transfer function of the form $\exp(-j^2/a^2)$, with $a = 2048$, and are subject to random noise of some sort at some level.

   Find a filter to apply to the data, and plot your best-guess reconstruction of the original uncorrupted dataset. Rather than attempting to characterize the noise in detail and applying an optimal filter as in *Numerical Recipes*, it will be sufficient simply to truncate the data in Fourier space at some frequency dictated by the form of the power spectrum you obtain. Turn in (i) your program, (ii) a graph of the untruncated power spectrum, clearly indicating where you chose to truncate (and why), and (iii) a plot of your final “uncorrupted” time sequence. Can you characterize the type of noise in the data?