

PHYS 501: Mathematical Physics I

Fall 2011

Homework #6

Solutions to questions 1 and 2

1. The inverse transform is

$$\begin{aligned} I &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin \omega a}{\omega} e^{-i\omega t} d\omega \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2i\omega} e^{i\omega(a-t)} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2i\omega} e^{-i\omega(a+t)} \\ &= I_1 - I_2. \end{aligned}$$

Both integrals have singularities at $\omega = 0$, on the integration path. We avoid them with small semicircular contours bypassing the singularity above, adding a contribution of $-\pi i$ times the relevant residue at $\omega = 0$. In all cases, the residue is $1/2i$, leading to a contribution of $-\frac{1}{2}$.

We close the contours with large semicircles at $|\omega| = \infty$. For integral I_1 , Jordan's lemma implies that we close the contour in $\text{Im}(\omega) > 0$ for $t < a$, and in $\text{Im}(\omega) < 0$ for $t > a$. Thus, for $t < a$ we have $I_1 - \frac{1}{2} = 0$, since no poles are enclosed by the chosen contour, so $I_1 = \frac{1}{2}$. For $t > a$, we have $I_1 - \frac{1}{2} = -1$ (negative since the contour is traversed in the clockwise direction), so $I_1 = -\frac{1}{2}$. Similarly, we have $I_2 = \frac{1}{2}$ for $t < -a$, $I_2 = -\frac{1}{2}$ for $t > -a$.

Combining these results, we have

$$I = I_1 - I_2 = \begin{cases} \frac{1}{2} - \frac{1}{2} & = 0, & t < -a, \\ \frac{1}{2} - \left(-\frac{1}{2}\right) & = 1, & -a < t < a, \\ -\frac{1}{2} - \left(-\frac{1}{2}\right) & = 0, & t > a. \end{cases}$$

For $|t| = a$, either $I_1 = 0$ or $I_2 = 0$ by symmetry. In either case we find $I = \frac{1}{2}$.

2. The equation to be solved is

$$y'' + xy = 0,$$

with the boundary condition that $|y|$ remain finite as $|x| \rightarrow \infty$.

(a) Qualitatively, for $x > 0$ we expect oscillatory behavior, analogous to the solution of $y'' + k^2y = 0$, except that the "frequency" k increases and the amplitude decreases with increasing x . We will demonstrate both these points explicitly in a moment. For $x < 0$ we expect exponential behavior, like the solution of $y'' - k^2y = 0$. In general there are exponentially growing and decaying solutions. The boundary condition selects the solution that goes (faster than) exponentially to zero as $x \rightarrow -\infty$.

(b) Let $Y(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} y(x) e^{i\omega x} dx$ be the Fourier transform of y . Then

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} x y(x) e^{i\omega x} dx = -iY'(\omega)$$

and the above equation transforms to

$$-\omega^2 Y - iY' = 0,$$

or

$$Y' = i\omega^2 Y,$$

the solution to which is

$$Y(\omega) = Y(0) e^{\frac{1}{3}i\omega^3}.$$

Transforming back, we find

$$y(x) = (2\pi)^{-1/2} Y(0) \int_{-\infty}^{\infty} e^{\frac{1}{3}i\omega^3 - i\omega x} d\omega.$$