PHYS 501: Mathematical Physics I

Fall 2011

Homework #3 solutions

1. (a) Writing $z = re^{i\theta}$ we see that $w = u + iv = r^{-1}e^{-i\theta}$. Hence the image of the line $\mathrm{Re}\, z = r\cos\theta = A$ is $u = r^{-1}\cos\theta = A^{-1}\cos^2\theta = \frac{1}{2}A^{-1}(1+\cos 2\theta)$, and $v = -r^{-1}\sin\theta = -\frac{1}{2}A^{-1}\sin 2\theta$. Thus $(u - \frac{1}{2}A^{-1})^2 + v^2 = \frac{1}{4}A^{-2}$, representing a circle centered on the real axis, at $w = \frac{1}{2}A^{-1}$, and passing through the origin. The image of $\mathrm{Re}\, z > A$ is the interior of the circle if $A > 0$, the exterior otherwise. Similarly, lines of constant Im z map to orthogonal circles centered on the imaginary axis.

   (b) Since $e^z = e^{x+iy}$, lines of constant $x = \mathrm{Re}\, z$ map to circles centered on the origin, of radius e^x. Lines of constant $y = \mathrm{Im}\ z$ map to straight lines through the origin, at an angle y to the real axis. Thus the region $\mathrm{Re}\, z > A$ maps to the exterior of a circle of radius e^A.

   (c) The function has three branch points, at $z = 0$ and $z = \pm i$. In order to prevent circulation around any branch point, branch cuts may (i) join all three points to infinity, (ii) join any two of the points to one another and the third to infinity, or (iii) join all three to one another and one of them to infinity. Any other combination (e.g. simply joining all three branch points) will still allow a contour to enlose all three, and the function will be multivalued along that contour.

2. (a) The function has poles at $z = \pm 1$, so we expect three different expansions in $s = z - 2$, for $|s| < 1$ (Taylor series), $1 < |s| < 3$, and $|s| > 3$ (Laurent series). Write $f(z) = \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right) = \frac{1}{2}\left(\frac{1}{1+s} + \frac{1}{3+s}\right)$ and construct Taylor or Laurent series in s using appropriate binomial expansions in each s range. For $|s| < 1$ expand both fractions as Taylor series in s. For $1 < |s| < 3$, again expand the second fraction as a Taylor series, and rewrite the first as $\frac{1}{s}\left(\frac{1}{1+s^{-1}}\right)$, where $|s^{-1}| < 1$, and expand as a series in s^{-1}. For $|s| > 3$ expand both fractions as series in s^{-1}. The solutions are

$$|s| < 1: \qquad f(z) = \sum_{n=0}^{\infty} \tfrac{1}{2}\,(1 + 3^{-n-1})\,(-s)^n \;=\; \tfrac{2}{3} - \tfrac{5}{9}s + \tfrac{14}{27}s^2 - \ldots$$

$$1 < |s| < 3: \qquad f(z) = \frac{1}{6}\sum_{n=0}^{\infty} \left(\frac{-s}{3}\right)^n + \frac{1}{2s}\sum_{n=0}^{\infty} (-s)^{-n}$$

$$|s| > 3: \qquad f(z) = \frac{1}{s}\sum_{n=0}^{\infty} \tfrac{1}{2}\,(1 + 3^n)\,(-s)^{-n} \;=\; s^{-1} - 2s^{-2} + 5s^{-3} - \ldots$$

   Note that the poles of f lie on the circles where the various expansions diverge.

   (b) (i) The Cauchy–Riemann conditions give $w(z) = e^z + c_1$, where c_1 is pure imaginary.

   (ii) Cauchy–Riemann again gives $w(z) = z^3 - z + c_2$, where c_2 is real.

   (iii) Inverting the relation $z = \tan w = \frac{\sin w}{\cos w} = -i\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}$, we obtain

$$\begin{aligned} w(z) &= \tfrac{1}{2}i\,\log\left(\frac{1-iz}{1+iz}\right) \;=\; -\tfrac{1}{2}\arg\left[\frac{1-iz}{1+iz}\right] + \tfrac{1}{2}i\,\log\left|\frac{1-iz}{1+iz}\right| \\ &= -\tfrac{1}{2}\left[\arg(1-iz) - \arg(1+iz)\right] + \tfrac{1}{4}i\,\log\left[\frac{x^2 + (1+y)^2}{x^2 + (1-y)^2}\right], \end{aligned}$$

so

$$
\begin{aligned}
u(x,y) &= \tfrac{1}{2}\left[\tan^{-1}\frac{x}{1+y} + \tan^{-1}\frac{x}{1-y}\right] = \tfrac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}\,, \\
v(x,y) &= \tfrac{1}{4}\log\left[\frac{x^2+(1+y)^2}{x^2+(1-y)^2}\right].
\end{aligned}
$$

3. (a) The integrand has a pole at $z = 0$, with residue 1, so the integral is $I = 2\pi i$.

   (b) The integrand has poles at $z = (n+\frac{1}{2})\pi$, but these lie outside the contour and so don't contribute. The integrand also has a removable singularity at $z = 0$ and hence can be made analytic there, so $I = 0$.

   (c) The integrand has a pole at $z = 1 - i$, which lies inside the circle, so the integral is $2\pi i$ times the residue there $(2 - 7i)$, and $I = 2\pi(7 + 2i)$.

   (d) The integrand has a pole of order 2 at $z = 2$, but this point lies outside the contour, so $I = 0$.

4. (a) Complete the contour with a large semicircle C_R of radius R in Im $z > 0$, argue that the integral along C_R is less than $\pi R/R^6 \to 0$ as $R \to \infty$, and evaluate the residue at the pole (of order 3) $z = i$ as $-3i/16$ to find $I = 3\pi/8$.

   (b) Write $I = \frac{1}{2}\,\mathrm{Im}\displaystyle\int_{-\infty}^{\infty}\frac{ze^{iz}}{z^2+1}\,dz$, complete the contour as in part (a), use Jordan's lemma to show that the integral around C_R goes to zero as $R \to \infty$, and evaluate the residue $(1/2e)$ at $z = i$ to find $I = \pi/2e$.

   (c) Write $I = \mathrm{Re}\displaystyle\int_{-\infty}^{\infty}\frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)}$, complete the contour as in parts (a) and (b), and as usual the integral around C_R goes to zero as $R \to \infty$. In this case, there are four poles, at $z = \pm ia$ and $\pm ib$, but only two of them lie inside the contour. The residue at ia is $\frac{ie^{-a}}{2a(a^2-b^2)}$, with similar expressions for the other residues. For any choice of a and b, the solution may be written as

$$
I = \frac{\pi}{b^2-a^2}\left[\frac{e^{-|a|}}{|a|} - \frac{e^{-|b|}}{|b|}\right].
$$

   (d) The integral is $I = \displaystyle\int_{-\infty}^{\infty}\frac{z^2}{2\cosh z}\,dz$. In this case, the usual trick of completing the contour with a large semicircle will not work, as the integrand has an infinite number of poles along the imaginary axis and the integral along the large semicircle does not go to zero. Instead, the contour to use is a rectangle having corners $-R$, R, $R+\pi i$, and $-R+\pi i$. This exploits the periodicity of $\cosh z$ in the imaginary direction. The sides parallel to the imaginary axis do not contribute as $R \to \infty$ and, using $\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y$, it follows that the integral along the top side is

$$
\begin{aligned}
\int_{\infty}^{-\infty}\frac{(x+i\pi)^2}{2\cosh(x+i\pi)}\,dx &= \int_{-\infty}^{\infty}\frac{(x^2-2i\pi x-\pi^2)}{2\cosh x}\,dx \\
&= I - \tfrac{1}{2}\pi^2\int_{-\infty}^{\infty}\frac{dz}{\cosh z}.
\end{aligned}
$$

   The second integral may be shown (by a repetition of the same approach, as done in class) to be π. Thus we have

$$
2I - \pi^3/2 = 2\pi i \times \mathrm{Res}(\pi i/2) \quad \Rightarrow \quad I = \pi^3/8\,.
$$