PHYS 431/750: Galactic Dynamics

Fall 2011

Solutions to Homework #5

1. (a) If $\rho(r) = Ar^{-2}$, we have $M(r) = 4\pi Ar$, and so $v_c^2 = GM(r)/r = 4\pi AG$. Hence

$$\rho(r) = \frac{v_c^2}{4\pi G r^2}.$$

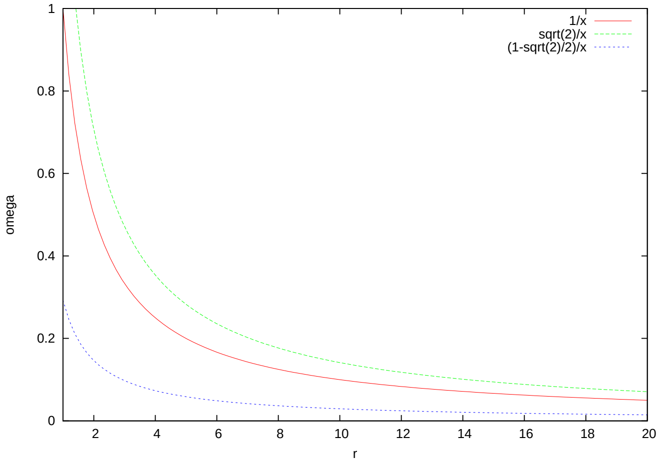
From Poisson's equation,

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} &= 4\pi G\rho = \frac{v_c^2}{r^2} \\ \Rightarrow \frac{d}{dr}r^2\frac{d\phi}{dr} &= v_c^2 \\ \Rightarrow r^2\frac{d\phi}{dr} &= v_c^2 r + \text{const} \\ \Rightarrow \phi &= v_c^2 \ln r + \text{const} \end{aligned}$$

where we have assumed that the force is regular at the origin to set the first constant equal to zero. Thus, in this case, we have

$$\begin{aligned} \Omega^2 &= \frac{v_c^2}{r^2} \\ \kappa^2 &= -2\frac{v_c^2}{r^2} + 4\frac{v_c^2}{r^2} = 2\Omega^2 \\ \Omega - \tfrac{1}{2}\kappa &= (1 - \tfrac{1}{2}\sqrt{2})\,\Omega, \end{aligned}$$

so all frequencies scale with Ω. The variation of Ω, κ, and $\Omega - \frac{1}{2}\kappa$ in this case are shown in the figure below.



(b) If the density is negligible, we can write $\nabla^2\phi = 0$. Hence, expanding in cylindrical polar coordinates, we have

$$\frac{\partial^2\phi}{\partial R^2} + \frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

so

$$\kappa^2 - \frac{3L_c^2}{R^4} + \Omega^2 + \nu^2 = 0.$$

where $R\,\Omega^2 = \partial\phi/\partial R$ and $L_c = R^2\,\Omega$. Thus

$$\kappa^2 + \nu^2 = \frac{3L_c^2}{R^4} - \Omega^2 = 2\Omega^2.$$

2. The shape (in plane polar coordinates R, ϕ) of an m-armed spiral is defined by

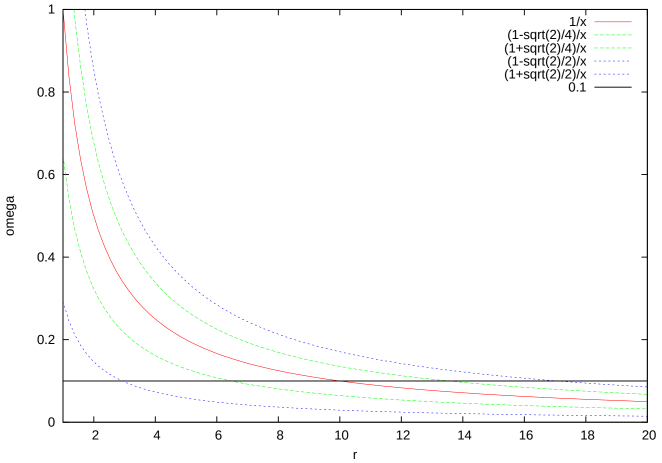
$$\cos\{m\,[\phi + f(R,t)]\} = 1,$$

where f is some arbitrary function. From Equation (5.8) of Sparke & Gallagher, we have

$$R\frac{\partial f}{\partial R} = \frac{1}{\tan\, i},$$

where i is the pitch angle. If i is constant, the solution to this equation is $f = \ln R/\tan i +$ constant, or $f\tan i = \ln R + k$. Moving outward at fixed ϕ from a point on an arm, the next arm we encounter will come when $m\,[\phi + f(R,t)]$ has increased by 2π, i.e. when f has increased by $2\pi/m$. For the logarithmic spiral, this means that $\ln R$ has increased by $2\pi\tan i/m$, so R increases by a factor of $\exp(2\pi\tan i/m)$.

3. From problem 1, $\kappa = \sqrt{2}\Omega \approx 36\,\mathrm{kms^{-1}kpc^{-1}}$ for $\Omega \approx 25\,\mathrm{kms^{-1}kpc^{-1}}$. The curves of Ω, $\Omega \pm \kappa/2$, and $\Omega \pm \kappa/4$, and the line $\Omega_p = \Omega(10)$ are shown below.



An m-armed spiral of patern speed Ω_p can persist only between the inner and outer Lindblad resonances, where $\Omega_p = \Omega - \kappa/m = \Omega(1 - \sqrt{2}/m)$ and $\Omega_p = \Omega + \kappa/m = \Omega(1 + \kappa/m)$, respectively. For $V_c(R) =$ constant, $\Omega \propto 1/R$, or

$$\frac{\Omega}{\Omega_p} = \frac{R_{CR}}{R},$$

where R_{CR} is the corotation radius. Hence the distance from the inner to the outer Lindblad resonance is

$$R_{OLR} - R_{ILR} = \frac{R_{CR}}{1-\sqrt{2}/m} - \frac{R_{CR}}{1+\sqrt{2}/m} = \frac{2\sqrt{2}\,m}{m^2-2} R_{CR}.$$

For fixed Ω_p (and hence the same R_{CR}), this distance is $2.83 R_{CR}$ for $m = 2$ and $0.94 R_{CR}$ for $m = 4$, smaller by a factor of 3.

4. (a) Differentiating the dispersion relation, we have

$$2\,(\omega - n\Omega)\,\frac{\partial \omega}{\partial k} = -2\pi G\Sigma\,\mathrm{sign}(k) + 2kv_s^2.$$

so

$$v_g = \frac{\partial \omega}{\partial k} = \mathrm{sign}(k)\,\frac{|k|v_s^2 - \pi G\Sigma}{\omega - n\Omega}.$$

(b) If $v_s\kappa = \pi G\Sigma$, the dispersion relation reduces to

$$|\omega - m\Omega| = |\kappa - |k|v_s|\,,$$

so

$$|v_g| = \frac{\big||k|v_s^2 - v_s\kappa\big|}{|\kappa - |k|v_s||} = |v_s|.$$