# PHYS 431/750: Galactic Dynamics

*Fall 2011*

Solutions to Homework #4

1. (a) In the frame of the massive body, the deflection angle for an encounter with impact parameter $b$ and velocity at infinity $V_\infty$ is $\theta_d = \pi - 2\theta_1$, where

$$\tan\theta_1 = \frac{bV_\infty^2}{GM}.$$

Note that, as $b$ drops from $\infty$ to 0, $\theta_1$ goes from $\pi/2$ to 0, and $\theta_d$ rises from 0 to $\pi$. Thus the the maximum possible deflection is $\pi$, corresponding to zero impact parameter. Such a deflection reverses the velocity vector of the incomer in this frame, producing a velocity change of $-2V_\infty$. The maximum relative velocity at infinity occurs when $v$ and $V$ are oppositely directed. In that case, $V_\infty = v + V$ and the final velocity of the light particle is $v - 2(v + V) = -v - 2V$, so the speed increases by $2V$.

(b) The mean square velocity is

$$\begin{aligned}\langle v^2\rangle &= \frac{\int_0^\infty v^2 f(v)\,dv}{\int_0^\infty f(v)\,dv} \\ &= \frac{\int_0^\infty v^4 e^{-v^2/a^2}\,dv}{\int_0^\infty v^2 e^{-v^2/a^2}\,dv} \\ &= \frac{\frac{3}{8}\pi^{1/2}a^5}{\frac{1}{4}\pi^{1/2}a^3} \\ &= \tfrac{3}{2}a^2,\end{aligned}$$

where $a^2 = 2kT/m$. Hence $\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT$, as claimed.

2. (a) In virial equilibrium, $E = \frac{1}{2}U \propto M^2/R$. Since energy is conserved by evaporation, $E =$ constant, so $R \propto M^2$.

(b) Since $R \propto M^2$ and $t_R \propto M^{1/2}R^{3/2}$, we can write

$$\frac{t_r}{t_{R0}} = \left(\frac{M}{M_0}\right)^{7/2}.$$

The mass of the cluster obeys the equation

$$\frac{dM}{dt} = -\frac{M}{\alpha t_R}.$$

Writing $m = M/M_0$ and $\tau = t/t_{R0}$, this becomes

$$\frac{dm}{d\tau} = -m^{-5/2}/\alpha,$$

the solution to which is

$$\tfrac{2}{7}(m^{7/2} - 1) = -\tau/\alpha,$$

or

$$m = \left(1 - \frac{7\tau}{2\alpha}\right)^{2/7}.$$

The lifetime of the cluster (when $m = 0$) is therefore $\tau = \frac{2}{7}\alpha$, or $t = \frac{2}{7}\alpha t_{R0}$. The mean density $\bar{\rho} \propto M/R^3 \propto M^{-5}$, so

$$\bar{\rho} = \bar{\rho}_0 \left(1 - \frac{7\tau}{2\alpha}\right)^{-10/7}.$$

(c) For $M_0 = 5 \times 10^5 M_\odot$, $R_0 = 10$ pc, and assuming a mean stellar mass of $0.5 M_\odot$, the initial relaxation time is

$$t_{R0} \sim \frac{N_0}{6\sqrt{2}\ln\Lambda}\left(\frac{GM_0}{R_0^3}\right)^{-1/2} = 13.4 \text{ Gyr},$$

where we have taken $\Lambda = \frac{1}{2}N_0$ and $N_0 = 10^6$. Hence, taking $\alpha = 136$, we find $t = 520$ Gyr.

3. We first note that the mass inside radius $r$ is

$$M(r) = \int_0^r 4\pi s^2 \rho(s) ds = \frac{v_c^2 r}{G},$$

so the circular velocity at radius $r$ is $v_c =$ constant.

(a) The magnitude of the acceleration due to dynamical friction (Sparke & Gallagher, Eq. 7.8) is

$$a_{df} = -\frac{4\pi G^2 M_s \rho(r) \ln\Lambda}{v^2} = \frac{GM_S \ln\Lambda}{r^2} \quad \text{if } v = v_c;$$

the direction is in the opposite direction to the velocity.

(b) Assuming that the satellite moves through a series of circular orbits, its angular momentum per unit mass at radius $r$ is

$$L(r) = v_c r.$$

(c) Since $v = v_c$ and the rate of change of angular momentum is equal to the applied torque, which is just $r a_{df}$ for a circular orbit, we have

$$\frac{dL}{dt} = v_c \frac{dr}{dt} = -\frac{GM_s \ln\Lambda}{r},$$

so

$$\frac{dr}{dt} = -\frac{GM_s \ln\Lambda}{v_c\, r}.$$

(d) The solution to this equation with $r = R$ at $t = 0$ is

$$R^2 - r^2 = \frac{2GM_s \ln\Lambda}{v_c} t,$$

so $r = 0$ when

$$t = \frac{v_c R^2}{2GM_s \ln\Lambda}.$$

(e) Evaluating this expression for $M_s = 2 \times 10^{10} M_\odot$, $R = 50$ kpc, $v_c = 220$ km/s, and $\Lambda = 20$, we have $t = 1.0$ Gyr.

4. For a distribution function $f(E)$, we have

$$\langle v_x^2\rangle = \frac{1}{n}\int dv_x dv_y dv_z\, v_x^2\, f[\phi + \tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2)],$$

while

$$\langle v_y^2\rangle = \frac{1}{n}\int dv_x dv_y dv_z\, v_y^2\, f[\phi + \tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2)].$$

Obviously these are the same, so it follows that $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle$ and the velocity distribution is isotropic.

5. (a) By definition, $\mathcal{E} = \psi - \frac{1}{2}v^2$, $v^2 = v_r^2 + v_t^2$, and $L = rv_t$. Writing $X = L^2$, we can express the volume element in velocity space, $d^3v = 2\pi v_t dv_t dv_r$, as

$$2\pi v_t dv_t dv_r = 2\pi v_t \begin{vmatrix} \frac{\partial\mathcal{E}}{\partial v_t} & \frac{\partial\mathcal{E}}{\partial v_r} \\ \frac{\partial X}{\partial v_t} & \frac{\partial X}{\partial v_r} \end{vmatrix}^{-1} d\mathcal{E}dX = 2\pi v_t \begin{vmatrix} -v_t & -v_r \\ 2r^2 v_t & 0 \end{vmatrix}^{-1} d\mathcal{E}dX = \frac{\pi d\mathcal{E}dX}{r^2 v_r}\,.$$

(b) For a distribution function

$$f(\mathcal{E}, L) = \begin{cases} A\delta(L^2)(\mathcal{E} - \mathcal{E}_0)^{-1/2} & (\mathcal{E} > \mathcal{E}_0) \\ 0 & (\mathcal{E} \le \mathcal{E}_0), \end{cases},$$

we have

$$\rho = \int f\, d^3v = A \int\!\!\int d\mathcal{E}dX \left(\frac{\pi}{r^2 v_r}\right) \delta(X)(\mathcal{E} - \mathcal{E}_0)^{-1/2}$$

Now write $v_r^2 = 2(\psi - \mathcal{E}) - X/r^2$, and do the $X$ integral to set $v_r^2 = 2(\psi - \mathcal{E})$, so

$$\rho(r) = \left(\frac{\pi A}{2\sqrt{2}}\right) r^{-2} \int_{\mathcal{E}_0}^{\psi(r)} (\mathcal{E} - \mathcal{E}_0)^{-1/2}(\psi - \mathcal{E})^{-1/2}\, d\mathcal{E}\,,$$

for $\psi(r) \ge \mathcal{E}_0$. By substituting $y = (\mathcal{E} - \mathcal{E}_0)/(\psi - \mathcal{E}_0)$, it is easily shown that the integral is simply $\int_0^1 y^{-1/2}(1-y)^{-1/2}\, dy = \pi$, which is independent of $\psi$ and hence of $r$, so

$$\rho(r) = Br^{-2}\,,$$

for $r < r_0$, where $\psi(r_0) = \mathcal{E}_0$ and $B = \pi^2 A/2\sqrt{2}$.