PHYS 431/531: Galactic Dynamics

Fall 2011

Solutions to Homework #3

1. (a) The Kuzmin potential is

$$\Phi_K(R,z) = -\frac{GM}{\sqrt{R^2 + (a+|z|)^2}}.$$

For $z > 0$ denote the denominator by $s = (R^2 + (a+z)^2)^{1/2}$. Then

$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial R^2} \\
&= GM\left[\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R^2}{s^3}\right) + \frac{\partial}{\partial z}\left(\frac{a+z}{s^3}\right)\right] \\
&= GM\left[\frac{2}{s^3} - \frac{3R^2}{s^5} + \frac{1}{s^3} - \frac{3(a+z)^2}{s^5}\right] \\
&= \frac{GM}{s^5}\left[3s^2 - 3\{R^2 + (a+z)^2\}\right] \\
&= 0,
\end{aligned}$$

and similarly for $z < 0$. Φ is not differentiable at $z = 0$; the potential represents a mass sheet in that plane.

(b) Let the surface density of the sheet be $\Sigma(R)$, and consider a "Gaussian pillbox"—a right cylinder of cross-sectional area δA and negligible extent in the z direction, with axis parallel to the z axis—straddling the sheet. Let $a_z(R,z) = -\partial\Phi\partial z$ be the z component of the accdeleration. Clearly $a_z(R,-z) = -a_z(R,z)$ and, in particular, the acceleration just below the disk is equal and opposite to the acceleration just above it, $a_z(R,0_-) = -a_z(R,0_+)$. Hence, by Gauss's law, $-2a_z(R,0_+)\delta A = 4\pi G\Sigma(R)\delta A$, so

$$\begin{aligned}
\Sigma(R) &= -\frac{1}{2\pi G}\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} \\
&= \left.\frac{aM}{2\pi}s^{-3/2}\right|_{z=0} \\
&= \frac{aM}{2\pi}(R^2 + a^2)^{-3/2}\,.
\end{aligned}$$

(c) The circular orbit speed is v_c, where

$$\begin{aligned}
v_c^2 &= -R\left.\frac{\partial\Phi}{\partial R}\right|_{z=0} \\
&= \left.\frac{GMR^2}{s^3}\right|_{z=0} \\
&= \frac{GMR^2}{(R^2+a^2)^{3/2}}\,.
\end{aligned}$$

2. Since r remains constant as v changes, and v_r^2 clearly is ≥ 0 after the velocity change, the old pericenter r_p still lies in the region where $v_r^2 \geq 0$, so the new pericenter r_p' must lie inside r_p.

3. (a) The pericenter separation is most easily determined by using the expressions for angular momentum and energy. At pericenter (separation r), the motion (speed v) is transverse, so we can write

$$L = bV = rv\,,$$
$$E = \tfrac{1}{2}V^2 = \tfrac{1}{2}v^2 - \frac{GM}{r}\,.$$

Eliminating $v = bV/r$ we find

$$V^2 r^2 + 2GMr - b^2V^2 = 0\,,$$

so

$$r = \frac{GM}{V^2}\left(\sqrt{1 + \frac{b^2V^4}{G^2M^2}} - 1\right)\,.$$

(b) The number of collisions per unit time is $n\sigma V$, where the cross section σ is given by

$$\sigma = \pi b^2 = \pi r^2\left(1 + \frac{2GM}{rV^2}\right)\,.$$

(The term in parentheses is the *gravitational focusing factor*—it determines the degree to which gravity increases the geometric cross section πr^2.)

4. (a) (i) Neglecting, as usual, the distinction betwen the half-mass radius and the virial radius, the virial theorem estimate of the mass is $M = 2R\langle v^2\rangle/G = 4.6 \times 10^5 M_\odot$. If we interpret the given 10 km/s as the 1-D (line of sight) velocity dispersion, the answer increases by a factor of 3, to $1.4 \times 10^6 M_\odot$.

(ii) Writing $\frac{1}{2}M = R\langle v^2\rangle/G = \frac{4\pi}{3}\langle m\rangle nR^3$, we find $R = \sqrt{\frac{3\langle v^2\rangle}{4\pi Gn\langle m\rangle}} = 1.7$ pc, so $M = 3100 M_\odot$.

(iii) For $t_D = 2\pi\left(\frac{GM}{R^3}\right)^{-1/2} = 10^6$ yr and $R = 1\,\text{pc}$, we have $M = 8800 M_\odot$. This answer depends particularly sensitively on the definition of t_D—if we omit the 2π, the mass becomes $M = 220 M_\odot$

(b) The relaxation time is

$$t_r = \frac{v^3}{8\pi G^2\langle m\rangle\rho\ln\Lambda},$$

where $v^2 = 3\sigma^2$. For the numbers in S&G Table 3.1, and $\langle m\rangle = 0.5 M_\odot$, we find $t_r = 32\,\text{Myr}$, $t_{cr} = 2r_c/\sigma = 0.089\,\text{Myr} = 2.8 \times 10^{-3}\,t_r$. For $\langle m\rangle = 0.3 M_\odot$, as in S&G, we find $t_r = 53\,\text{Myr}$, $t_{cr} = 1.7 \times 10^{-3}\,t_r$. Note that we can't use the result $t_R \sim (N/\ln\Lambda)t_{cr}$ here, since that refers only to global averages.

5. (a) For cylindrical, axisymmetric geometry, we expect the potential ϕ and gravitational acceleration $\mathbf{a} = -\nabla\phi$ at any point to be functions of R (distance from the axis) only. Applying Gauss's law to Poisson's equation in a cylinder (C, surface S) of length H and radius R, coaxial with the fluid, we have

$$\int\int\int_C 4\pi G\rho_0\, d^3V = \int\int\int_C \nabla^2\phi\, d^3V = \int\int_S \nabla\phi \cdot d^2\mathbf{S} = -2\pi RHa(R)\,.$$

Hence, since ρ_0 is constant, we have

$$4\pi G\rho_0\pi R^2 H = -2\pi RHa,$$

so

$$\mathbf{a}(R) = -2\pi G\rho_0\,\mathbf{R} = -2\pi G\rho_0\,(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}),$$

where x and y are coordinates in the plane perpenidcular to the axis.

(b) Setting $\mathbf{v} = \mathbf{0}$, $\partial/\partial t = 0$, and $\nabla P = 0$, in Euler's equation

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla P}{\rho} - \nabla\phi - 2\mathbf{\Omega}\times\mathbf{v} + \Omega^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})$$

yields

$$\mathbf{0} = -\nabla\phi + \Omega^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}).$$

Substituting $\nabla\phi$ from part (a), we find

$$\Omega^2 = 2\pi G\rho_0$$

as the condition for equilibrium.

(c) Now for the hard part! Expand, as before, $P = P_0 + P_1$, $\rho = \rho_0 + \rho_1$, $\mathbf{v} = \mathbf{v}_1$, where all subscript "1" quantities are small and $P_1 = v_s^2\rho_1$. The continuity equation and Euler's and Poisson's equations become

$$\begin{aligned} \frac{\partial\rho_1}{\partial t} + \rho_0\nabla\cdot\mathbf{v}_1 &= 0 \\ \frac{\partial\mathbf{v}_1}{\partial t} &= -\frac{\nabla P_1}{\rho_0} - \nabla\phi_1 - 2\mathbf{\Omega}\times\mathbf{v}_1 \\ \nabla^2\phi_1 &= 4\pi G\rho_1\,. \end{aligned}$$

Recall that $\mathbf{\Omega} = \Omega\,\hat{\mathbf{z}}$ and seek solutions with behavior $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, to find

$$\begin{aligned} -i\omega\rho_1 + i\rho_0\mathbf{k}\cdot\mathbf{v} &= 0 \\ -i\omega\mathbf{v}_1 &= -i\mathbf{k}\frac{P_1}{\rho_0} - i\mathbf{k}\phi_1 - 2\Omega(-v_{1y}\hat{\mathbf{x}} + v_{1x}\hat{\mathbf{y}}) \\ -k^2\phi_1 &= 4\pi G\rho_1\,, \end{aligned}$$

where $\mathbf{v}_1$, ρ_1, and ϕ_1 are now wave amplitudes. Eliminating ρ_1 and ϕ_1 by substituting the first and third equations into the second, we find

$$\omega\mathbf{v}_1 = \left(\frac{v_s^2}{\omega} - \frac{4\pi G\rho_0}{\omega k^2}\right)(\mathbf{k}\cdot\mathbf{v})\mathbf{k} - 2i\Omega(-v_{1y}\hat{\mathbf{x}} + v_{1x}\hat{\mathbf{y}}),$$

or, in components (and dropping the "1" subscript),

$$\begin{aligned}
\omega v_x - \alpha k_x^2 v_x - \alpha k_x k_y v_y - \alpha k_x k_z v_z - 2i\Omega v_y &= 0 \\
\omega v_y - \alpha k_x k_y v_x - \alpha k_y^2 v_y - \alpha k_y k_z v_z + 2i\Omega v_x &= 0 \\
\omega v_z - \alpha k_z k_x v_x - \alpha k_z k_y v_y - \alpha k_z^2 v_z &= 0\,,
\end{aligned}$$

where

$$\alpha = \left(\frac{v_s^2}{\omega} - \frac{4\pi G \rho_0}{\omega k^2} \right).$$

For this equation to have a solution we require

$$\begin{vmatrix} \omega - \alpha k_x^2 & -\alpha k_x k_y - 2i\Omega & -\alpha k_x k_z \\ -\alpha k_x k_y + 2i\Omega & \omega - \alpha k_y^2 & -\alpha k_y k_z \\ -\alpha k_x k_z & -\alpha k_y k_z & \omega - \alpha k_z^2 \end{vmatrix} = 0.$$

The two cases to be considered are

(i) $k_x = k_y = 0$, so

$$\begin{aligned}
&\begin{vmatrix} \omega & -2i\Omega & 0 \\ 2i\Omega & \omega & 0 \\ 0 & 0 & \omega - \alpha k_z^2 \end{vmatrix} = 0 \\
\Rightarrow\quad & (\omega^2 - 4\Omega^2)(\omega - \alpha k_z^2) \quad = 0 \\
\Rightarrow\quad & \omega = 2\Omega \quad \text{or} \quad \omega = \alpha k_z^2\,.
\end{aligned}$$

Substituting for α, the latter solution becomes

$$\omega^2 = v_s^2 k_z^2 - 4\pi G \rho_0\,,$$

which is the usual Jeans condition.

(ii) $k_z = 0$, so

$$\begin{aligned}
&\begin{vmatrix} \omega - \alpha k_x^2 & -\alpha k_x k_y - 2i\Omega & 0 \\ -\alpha k_x k_y + 2i\Omega & \omega - \alpha k_y^2 & 0 \\ 0 & 0 & \omega \end{vmatrix} = 0 \\
\Rightarrow\quad & \omega(\omega^2 - \alpha\omega k^2 - 4\Omega^2) = 0 \\
\Rightarrow\quad & \omega = 0 \quad \text{or} \quad \omega^2 = v_s^2 k^2 - 4\pi G \rho_0 + 4\Omega^2\,.
\end{aligned}$$

Note that, since $\Omega^2 = 2\pi G \rho_0$, the nonzero solution has $\omega^2 = v_s^2 + 4\pi G \rho_0 > 0$, so the system is always stable.