PHYS 431/531: Galactic Dynamics

Fall 2011, Homework #3

(Due November 10, 2011)

1. (a) Verify that the Kuzmin potential

$$\Phi_K(R,z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}}$$

has $\nabla^2\Phi = 0$ for $z \neq 0$, and so represents a surface density distribution $\Sigma(R)$ in the plane $z = 0$.

(b) Use Gauss's law to determine $\Sigma(R)$.

(c) What is the circular orbit speed for a particle moving in the plane of the disk?

2. A star orbiting in a spherical potential (*not* necessarily Keplerian) suffers an arbitrary instantaneous velocity change while it is at pericenter. Show that the pericenter distance of the new orbit cannot be larger than that of the original orbit.

3. (a) A test particle of negligible mass approaches a star of mass M on an unbound orbit with velocity at infinity V and impact parameter b. By considering the particle's (conserved) energy and angular momentum, as in class, derive an equation relating its distance of closest approach to the star, r_p, to M, V, and b. The quantity πb^2 is the star's *cross section* for an approach within distance r_p.

(b) Now imagine that the star has radius R and is moving with velocity V through a uniform "sea" of such test particles, and with number density n. Use the result of part (a) to estimate the rate at which test particles strike the star.

4. (a) Estimate the masses of star clusters having (i) root mean square velocity 10 km/s and half-mass radius 10 pc, (ii) mean density 100 pc^{-3}, rms velocity 2 km/s, and mean stellar mass $0.8M_\odot$, (iii) dynamical time 10^6 yr and radius 1 pc.

(b) Assuming an average stellar mass of $0.5M_\odot$ and taking $\Lambda = r_c/1\,\mathrm{AU}$, use the information in Sparke & Gallagher Table 3.1 to find the relaxation time t_r at the center of the globular cluster 47 Tucanae. Show that the crossing $t_\mathrm{cr} \approx 2r_c/\sigma_r \sim 10^{-3}t_\mathrm{r}$.

5. *[Graduate students only]* Consider a homogeneous self-gravitating fluid of uniform density ρ_0 contained within a rotating cylinder of radius R_0. The cylinder and the fluid rotate at angular speed Ω about the axis of the cylinder, which we take to be the z-axis, so $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$.

(a) Show that the gravitational force per unit mass at distance r from the axis is

$$-\nabla\phi = -2\pi G\rho_0(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}),$$

directed radially toward the axis and perpendicular to it.

(b) Euler's equation for the fluid in the rotating frame is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{\rho} - \nabla \phi - 2\boldsymbol{\Omega} \times \mathbf{v} + \Omega^2 (x\hat{\mathbf{x}} + y\hat{\mathbf{y}})\,.$$

Find the condition on Ω such that the fluid is in equilibrium with no pressure gradients (i.e. no Jeans swindle).

(c) Now, still in the rotating frame, consider small perturbations about this equilibrium, analogous to the homogeneous 3-D fluid discussed in class. Assume an equation of state $P = v_s^2 \rho$. Find the dispersion relation $\omega^2(k)$ for waves propagating (i) perpendicular and (ii) parallel to $\boldsymbol{\Omega}$. Under what circumstances is each class of wave stable?