PHYS 431/531: Galactic Dynamics

Fall 2011

Solutions to Homework #2

1. (a) From the definitions of A and B,

$$A \equiv -\tfrac{1}{2}R\left(\frac{V}{R}\right)' = -\tfrac{1}{2}V' + \tfrac{1}{2}\frac{V}{R}$$
$$B \equiv -\tfrac{1}{2}\frac{(RV)'}{R} = -\tfrac{1}{2}V' - \tfrac{1}{2}\frac{V}{R}.$$

Hence $A + B = -V'$, $A - B = V/R$.

(b) For

$$\rho(R) = \rho_0\left(1 + \frac{R^2}{a^2}\right)^{-1},$$

the mass inside radius R is

$$M(R) = \int_0^R 4\pi R^2 \rho(R)\, dR = 4\pi\rho_0 a^3\left[\frac{R}{a} - \tan^{-1}\frac{R}{a}\right].$$

The squared circular orbital speed then is

$$V^2(R) = \frac{GM(R)}{R} = 4\pi G\rho_0 a^2\left[1 - \frac{a}{R}\tan^{-1}\frac{R}{a}\right],$$

from which it follows that

$$VV' = \frac{2\pi G\rho_0 a^3}{R^2}\left[\tan^{-1}\frac{R}{a} - \frac{R/a}{1 + R^2/a^2}\right].$$

The expressions for A and B follow, but are messy and not very edifying. Looking at the limit $R \gg a$, using the fact that $\tan^{-1}x \approx \frac{1}{2}\pi - 1/x$ for large x, and retaining only leading terms, we have

$$V \approx \sqrt{4\pi G\rho_0 a^2}\left[1 - \frac{\pi a}{4R}\right], \quad VV' \approx \frac{\pi^2 G\rho_0 a^3}{R^2}\left[1 - \frac{4a}{\pi R}\right],$$

so

$$A \approx -B \approx \tfrac{1}{2}\frac{V}{R} \approx \sqrt{\pi G\rho_0}\,\frac{a}{R}, \quad V' \approx \frac{\pi}{2}\sqrt{\pi G\rho_0}\,\frac{a^2}{R^2} \ll \frac{V}{R}.$$

2. (a) The Jeans length and mass are given by

$$\lambda_J^2 = \frac{\pi v_s^2}{G\rho}, \quad M_J = \frac{4\pi}{3}\rho\left(\tfrac{1}{2}\lambda_J\right)^3,$$

where the density here is $\rho = 1.2$ kg/m^3 and the sound speed is $v_s = 330$ m/s. Hence $\lambda_J = 6.54 \times 10^7$ m $= 65,400$ km, $M_J = 1.76 \times 10^{23}$ kg $= 8.78 \times 10^{-8} M_\odot$.

(b) The dispersion relation is

$$\omega^2 = v_s^2 k^2 - 4\pi G\rho.$$

For a sound wave with $\lambda = 1\,\mathrm{m}$, $k = 2\pi\,\mathrm{m}^{-1}$ and the two terms on the right-hand side of the above relation are $w_0^2 = 4.3 \times 10^6$ and 1.0×10^{-9}, respectively, where the zero-gravity angular frequency is $\omega_0 = v_s k = 2.01 \times 10^3\,\mathrm{s}^{-1}$. The change in frequency is

$$\delta\omega \;=\; \sqrt{\omega_0^2 - 4\pi G\rho} - \omega_0 \;\approx\; -\frac{2\pi G\rho}{\omega_0} = -2.43 \times 10^{-13}\,\mathrm{s}^{-1},$$

so $\delta f = -\delta\omega/2\pi = -3.86 \times 10^{-14}$ Hz.

(c) If $\lambda = \frac{1}{2}\lambda_J$, then $k^2 = 4k_J^2 = 16\pi G\rho/v_s^2$. Substituting this value of k into the above dispersion relation, we find

$$\frac{\omega^2}{v_s^2 k^2} = 1 - \frac{4\pi G\rho}{v_s^2 k^2} = \frac{3}{4}.$$

Hence $\omega = \sqrt{\frac{3}{4}}\, v_s k$, $f = \sqrt{\frac{3}{4}}\, v_s/\lambda = 0.866\, v_s/\lambda$, and $\delta f/f = 0.134$.

3. (a) For a particle moving within a homogeneous sphere of mass M and radius a, the mass inside radius r is $m(r) = \frac{4}{3}\pi M(r/a)^3$, so the equation of motion is

$$\ddot{r} + \frac{GM}{a^3} r = 0\,.$$

The solution satisfying the initial condition $r = a, \dot{r} = 0$ is

$$r = a\cos\Omega t\,,$$

where $\Omega^2 = GM/a^3$. Obviously $r = 0$ at time $t = t_1 = \pi/2\Omega = \frac{\pi}{2}\sqrt{\frac{a^3}{GM}} = \sqrt{\frac{3\pi}{16G\rho}}$.

(b) If instead the entire sphere is collapsing, then for a point on the surface, $m(r) = M$, so

$$\ddot{r} + \frac{GM}{r^2} = 0\,.$$

Writing the energy per unit mass as $E = \frac{1}{2}\dot{r}^2 - GM/r = -GM/a$, we have $\dot{r}^2 = 2GM\left(\frac{1}{r} - \frac{1}{a}\right)$, so the collapse time is

$$t_2 = \sqrt{\frac{a}{2GM}}\int_0^a \frac{dr}{\sqrt{a/r - 1}}\,.$$

Substituting $r = a\sin^2\theta$, we find

$$t_2 = 2a\sqrt{\frac{a}{2GM}}\int_0^{\pi/2} \sin^2\theta d\theta = \frac{\pi}{2}\sqrt{\frac{a^3}{2GM}} = t_1/\sqrt{2}\,.$$

4. In virial equilibrium, the (3-D) rms velocity of stars is

$$\langle v^2\rangle = \frac{GM}{2R}\,,$$

so for $R = 100$ kpc and $M = 10^{12}$ solar masses, we have $\langle v^2\rangle^{1/2} = 147$ km/s. If this speed is representative of the gas, with $\frac{3}{2}kT = \frac{1}{2}m\langle v^2\rangle$, the corresponding temperature is

$$T = \frac{m\langle v^2\rangle}{3k} = \frac{GMm}{6kR} = 8.7 \times 10^5 K.$$