PHYS 431/531: Galactic Dynamics

Fall 2011

Solutions to Homework #1

1. (a) Stars of mass $2M_\odot$ have lifetimes of $\tau = 1.25$ Gyr, so at an age of $t_0 = 10$ Gyr all $2M_\odot$ stars born within the past 1.25 Gyr are still around today. Hence, the survivor fraction is

$$f = \frac{\int_{t_0-\tau}^{t_0} e^{-t/T} dt}{\int_0^{t_0} e^{-t/T} dt} = \frac{e^{\tau/T} - 1}{e^{t_0/T} - 1} = 1.9 \times 10^{-2}.$$

(b) Stars of mass $5M_\odot$ have $\tau = 80$ Myr, so $f = 1.0 \times 10^{-3}$.

2. (a) Apparent magnitude m depends on luminosity L according to $m = -2.5\log_{10} L + \text{const}$, where the constant includes both distance and calibration terms. For an unresolved binary of identical stars, the luminosity is $2L$, so the apparent magnitude is $m - 2.5\log_{10} 2 = m - 0.753$.

(b) The star has $m_V = 10$ and, according to Sparke & Gallagher Table 1.4, the absolute visual magnitude of an A0 main sequence star is $M_V = 0.8$. Since $m - M = 5\log_{10}(d/10\,\text{pc})$, it follows that the distance $d = 692$ pc.

3. (a) The stellar mass function is $dN/dm = Am^{-\alpha}$, for $0.2M_\odot = M_l < m < M_u = 100M_\odot$ and $\alpha = 2.35$. The number of stars having masses less than m is

$$N(m) = \int_{M_l}^{m} dN = \frac{A}{1-\alpha}\left[m^{1-\alpha}\right]_{M_l}^{m} = \frac{A}{\alpha - 1}\left(M_l^{1-\alpha} - m^{1-\alpha}\right).$$

The total number of stars is $N_{tot} = N(M_u) \approx AM_l^{1-\alpha}/(\alpha - 1)$ for $M_l \ll M_u$, so the lower mass limit effectively determines the number of stars. Substituting for A we have

$$N(m) = N_{tot}\left[\frac{1 - \left(\frac{m}{M_l}\right)^{1-\alpha}}{1 - \left(\frac{M_u}{M_l}\right)^{1-\alpha}}\right],$$

which equals $\frac{1}{2}N_{tot}$ when $m = 1.7M_l = 0.33M_\odot$.

Similarly, the total mass in stars less massive than m is

$$M(m) = \int_{M_l}^{m} m dN = M_{tot}\left[\frac{1 - \left(\frac{m}{M_l}\right)^{2-\alpha}}{1 - \left(\frac{M_u}{M_l}\right)^{2-\alpha}}\right],$$

so the lower limit also dominates here (but barely, in this case—the second term in the denominator is roughly 0.11), and half the total mass is in stars having mass less than $M_1 = 5.33M_l = 1.07M_\odot$. Note that, if the second term in the denominator is neglected, the result becomes $M_1 = 7.25M_l = 1.45M_\odot$.

The total luminosity of stars less massive than m is

$$L(m) \propto \int_{M_l}^{m} m^4 dN = L_{tot} \left[\frac{1 - \left(\frac{m}{M_l}\right)^{5-\alpha}}{1 - \left(\frac{M_u}{M_l}\right)^{5-\alpha}} \right].$$

The total luminosity is $L_{tot} \approx A M_u^{5-\alpha}/(5-\alpha)$, so the massive stars dominate. Half of the total luminosity is accounted for by stars having masses greater than $M_2 = 0.77 M_u = 77 M_\odot$.

(b) Write the Kroupa mass function as

$$\xi(M) = \begin{cases} C\,M^{-\gamma} & (M < M_C) \\ B\,M^{-\beta} & (M_C < M < M_B) \\ A\,M^{-\alpha} & (M_B < M < M_U) \end{cases},$$

where $M_C = 0.1 M_\odot$, $M_B = 0.5 M_\odot$, $M_U = 100 M_\odot$, $\gamma = 0.3$, $\beta = 1.3$, and $\alpha = 2.35$.

A is the same as before, and we can arbitrarily set it to 1 and work in solar masses throughout. The continuity of ξ at M_C and M_B implies

$$B = A\,M_B^{\beta-\alpha} = 2.071, \qquad C = B\,M_C^{\gamma-\beta} = 20.71.$$

The Kroupa and Salpeter distributions are identical for $M > M_B = 0.5$, so we can confine our attention to stars with $M < M_B$. For the Salpeter power-law, the number of stars with mass between M and M_B is

$$N_P(M) = \frac{1}{\alpha - 1}\left[M^{1-\alpha} - M_B^{1-\alpha}\right] = 0.7407 M^{-1.35} - 1.8882.$$

For the Kroupa distribution, the total number of stars with $M < M_B$ is

$$\begin{aligned} N_K &= \int_0^{M_C} C M^{-\gamma} dM + \int_{M_C}^{M_B} B M^{-\beta} dM \\ &= \frac{C}{1-\gamma} M_C^{1-\gamma} + \frac{B}{\beta - 1}\left[M_C^{1-\beta} - M_B^{1-\beta}\right] \\ &= 11.178. \end{aligned}$$

Hence the power-law and Kroupa results are the same for $M = 0.119\,M_\odot$.

Similarly, for the total mass,

$$M_P(M) = \frac{1}{\alpha - 2}\left[M^{2-\alpha} - M_B^{2-\alpha}\right] = 2.857\,M^{-0.35} - 3.641,$$

while

$$\begin{aligned} M_K &= \int_0^{M_C} C M^{1-\gamma} dM + \int_{M_C}^{M_B} B M^{1-\beta} dM \\ &= \frac{C}{2-\gamma} M_C^{2-\gamma} + \frac{B}{2-\beta}\left[M_B^{2-\beta} - M_C^{2-\beta}\right] \\ &= 1.474, \end{aligned}$$

so the power-law and Kroupa results agree for $M = 0.189\,M_\odot$.

Finally, for the total luminosity,

$$L_P(M) = \frac{1}{5-\alpha}\left[M_B^{5-\alpha} - M^{5-\alpha}\right] = 0.06012 - 0.3774M^{2.65},$$

while

$$\begin{aligned} L_K &= \int_0^{M_C} CM^{4-\gamma}dM + \int_{M_C}^{M_B} BM^{4-\beta}dM \\ &= \frac{C}{5-\gamma}M_C^{5-\gamma} + \frac{B}{5-\beta}\left[M_B^{5-\beta} - M_C^{5-\beta}\right] \\ &= 0.04305, \end{aligned}$$

so the power-law and Kroupa results agree for $M = 0.311\,M_\odot$.

4. (a) Assume that the sheet lies in the $x-y$ plane. Obviously by symmetry the acceleration is in the z direction and is independent of x and y. Write it as $a(z)$, where (again by symmetry) $a(-z) = -a(z)$. Now apply Gauss's law ($\int_V \nabla^2\phi = \int_S \nabla\phi \cdot d\mathbf{S}$) to a right cylinder with axis in the z direction, of cross-sectional area δA, and extending from $-z$ to $+z$. Only the ends of the cylinder contribute to the total flux, and the mass inside is $\Sigma\delta A$. Thus,

$$-2a(z)\delta A = 4\pi G\Sigma\delta A\,,$$

so $a(z) = -2\pi G\Sigma$ (for $z > 0$), independent of z.

(b) The maximum distance above the plane is $z_{max} = v_0^2/2|a|$, where v_0 is the z velocity as the star crosses the plane. Hence

$$|a| = v_0^2/2z_{max} = 2\pi G\Sigma\,,$$

so

$$\Sigma = v_0^2/4\pi G z_{max} = 0.070\,\mathrm{kg/m}^2 = 33\,M_\odot/\mathrm{pc}^2\,.$$