Properties of the real numbers

Drexel University
PGSA Informal Talks

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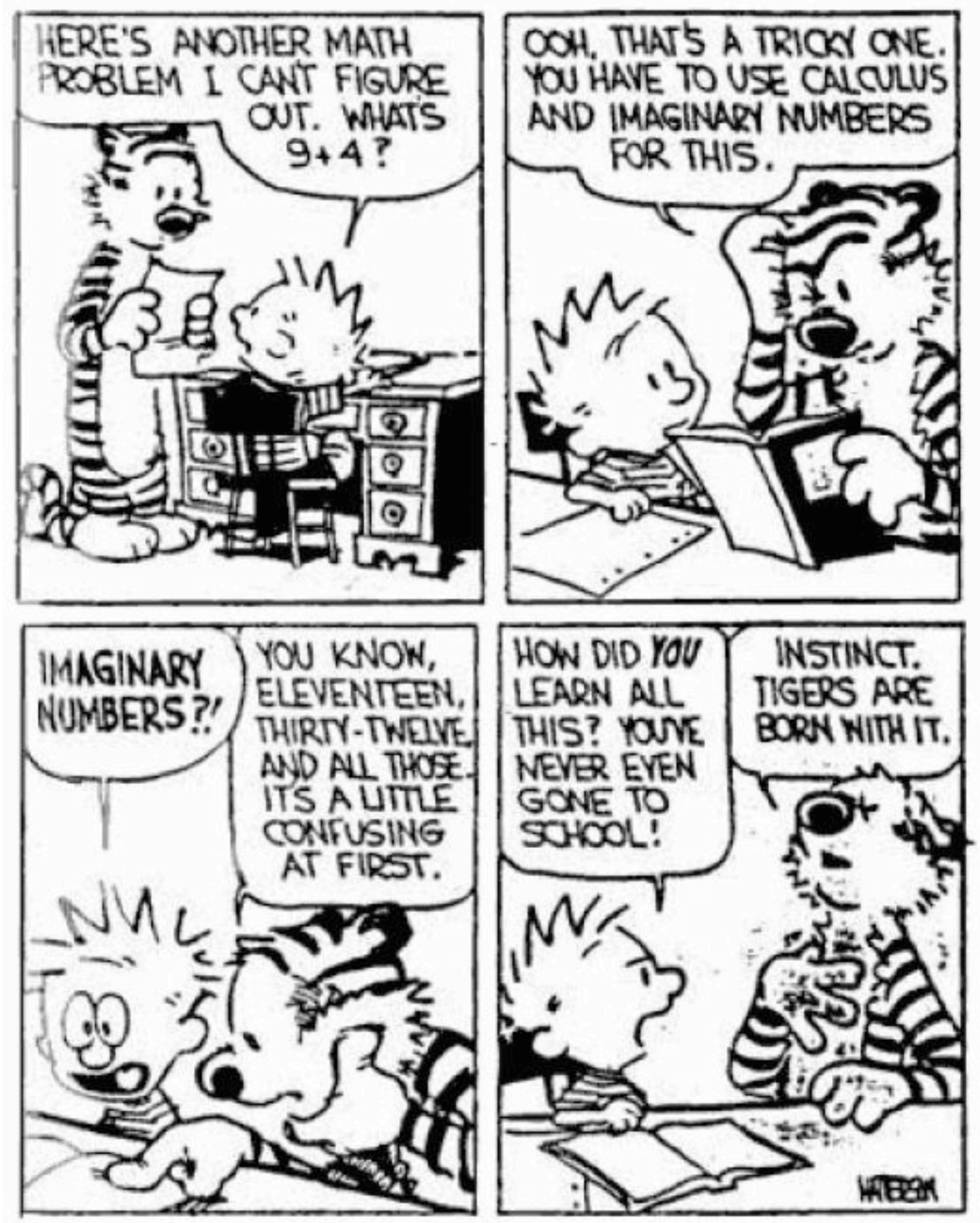
January 2012



★ We've *used* the real number line since elementary school, but that's not the same as *defining* it.

★ We'll review the *synthetic approach* with Hilbert's definition of $\mathbb{R}$ as the **complete ordered field**.

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Confession: I was an undergrad mathematics major.

Why bother?

★ Physical theories often require explicit or implicit assumptions about real numbers.

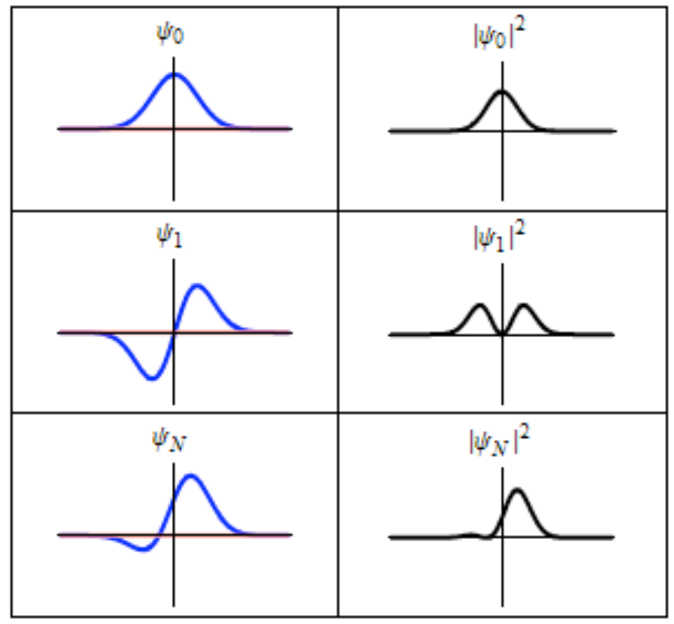
★ Quantum mechanics assumes that physical measurement results are eigenvalues of a Hermitian operator, which must be real numbers.

State vectors and wavefunctions can be complex. Are they physically “real”?

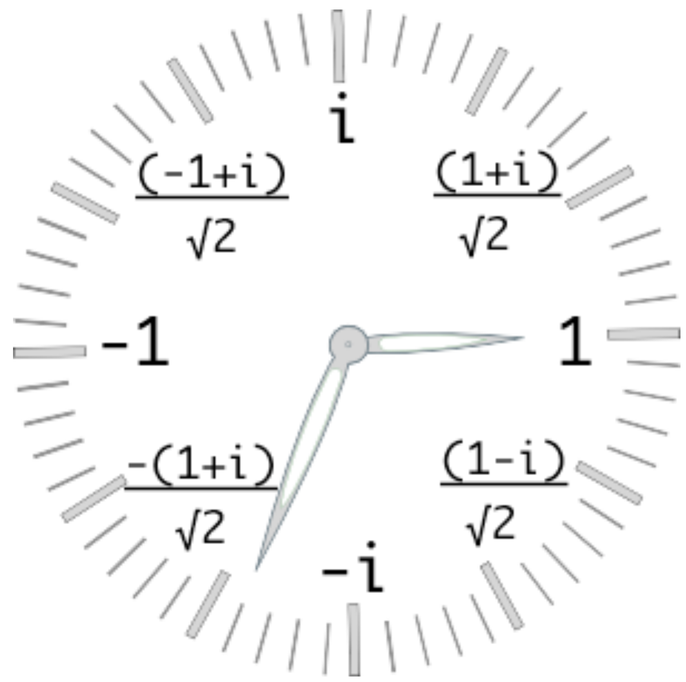
★ Special relativity can use *imaginary time*, which is related to *Wick rotation* in some quantum gravity theories. Is imaginary time unphysical?

(SR can also be described without imaginary time. Misner, Thorne & Wheeler recommend <u>never</u> using ict as a time coordinate.)

★ Is spacetime discrete or continuous? Questions like these require a rigorous definition of “real.”



wavefunctions and magnitudes



imaginary time?

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Spin networks in loop quantum gravity assume discrete time evolution.

Mini-biography of David Hilbert

★ First to use phrase “complete ordered field.”

★ Published Einstein’s GR equation before Einstein!
(For details, look up Einstein-Hilbert action.)

★ Chairman of the Göttingen mathematics department, which had enormous influence on modern physics.
(Born, Landau, Minkowski, Noether, von Neumann, Wigner, Weyl, et al.)

★ Ahead of his time re: racism, sexism, nationalism.

“Mathematics knows no races or geographic boundaries.”

“We are a university, not a bath house.” (in support of hiring Noether)

Minister Rust: "How is mathematics in Göttingen now that it has been freed of the Jewish influence?” Hilbert: “There is really none any more.”

★ Hilbert-style *formalism* (my paraphrasing):

1. State your assumptions explicitly.
2. Define things by what you can do with them.
3. Be logically consistent; avoid contradictions.

David Hilbert in 1912

Hilbert’s tombstone

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Hilbert was mostly self-taught at physics: “Physics is becoming too difficult for the physicists.”

- Noether used and extended Hilbert’s methods for rings, fields, and group invariants.
- Hilbert often scheduled her lectures under his name. She was later a salaried “temporary lecturer.”
- Mention Noether’s theorem: differentiable symmetry of action -> conservation law.

Hilbert retired in 1930; banquet with Bernhard Rust was 1933. Rust: “The whole function of education is to create Nazis.”

Number fields (formal definition)

★ A **field** is a set *R* with closed binary operators + and $*$ obeying these rules:

1. Associative addition $(a+b)+c=a+(b+c)$
2. Commutative addition $a+b=b+a$
3. Additive identity $\exists 0 : a+0=a \ \forall a$
4. Additive inverses $\forall a, \ \exists -a : -a+a=0$
5. Associative multiplication $(a*b)*c=a*(b*c)$
6. Distributive law $a*(b+c)=a*b+a*c$
7. Commutative multiplication $a*b=b*a$
8. Multiplicative identity $\exists 1 : a*1=a \ \forall a$
9. Multiplicative inverses $\forall a \neq 0, \ \exists \frac{1}{a} : \frac{1}{a}*a=1$

★ A **ring** needs only obey the first 6 axioms. A **division ring** obeys all but Rule 7.

★ This is a lot of stuff to memorize. A more intuitive approach would be nice.

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Number field is my term. The word “field” is used for many things in physics.

Number fields (lazy definition)

★ A **field** is a bunch of numbers with some logical rules for doing arithmetic.

★ **Arithmetic** is anything you can do with these symbols: $+ \; - \; * \; \div$

(Important Exception: dividing by zero is *verboten*.)

★ The number field rules are designed to make $+ \; - \; * \; \div$ behave themselves according to what we learned in elementary school.

★ A **ring** is like a field, except it can misbehave by breaking rules 7, 8, or 9:

¬ 7: Multiplication might not commute.
¬ 8: The number 1 might be missing.
¬ 9: Some numbers might not have reciprocals.

★ Because rings don't obey Rule 9, they can have **zero divisors.** Notorious example: real *n* x *n* matrices form a ring with singular matrices as zero divisors.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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Division rings have no zero divisors.

Division rings are sometimes called “skew fields.”

Constructing the rational numbers

★ Name your fingers 1 2 3 4 5 6 7 8 9 10.
Define + by counting with a friend. Define $*$ by composition of addition.
Problem: + and $*$ are not closed. (Not enough fingers!)

★ Define the **natural numbers** $\mathbb{N}$ by inventing "hypothetical fingers."
This is an *additive semigroup*: + is closed and associative.
Problem: There is no additive identity.

★ Define the **whole numbers** $\mathbb{W}$ by inventing the number 0.
This is an *additive monoid*: + is closed, associative, and has an identity.
Problem: There are not enough additive inverses.

★ Define the **integers** $\mathbb{Z}$ by inventing negative numbers.
This is an *additive group*: + is closed, associative, and invertible.
Problem: There are not enough multiplicative inverses.

★ Define the **rationals** $\mathbb{Q}$ by inventing fractions.
This is a number field: arithmetic works! (Don't divide by zero.)
Problem: Some numbers are still missing. For example, what is $\sqrt{2}$?

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Roots of rational polynomials are **algebraic** numbers; the other reals are **transcendental**. (Next slide.)

Some definitions of "complete"

★ **Complete** (Hilbert): Every *totally ordered* field is a subset of $\mathbb{R}$.

★ **Complete** (metric topology): Every Cauchy sequence converges in $\mathbb{R}$.
A sequence is **Cauchy** iff its terms eventually become arbitrarily close together.
(This requires a definition of *metric space*, which depends on the definition of $\mathbb{R}$.)

$$e = \lim_{N\to\infty}\left(\sum_{k=0}^{N}\frac{1}{k!}\right) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$$

Each choice of N defines a partial sum S_N. The sequence $\{S_1, S_2 \ldots\}$ is Cauchy.

We can use Cauchy sequences to define **transcendental reals** like *e* and π.
These numbers are neither rational nor roots of rational polynomials.

★ If an *inner product* is defined on a *vector space V*, then its *natural norm* defines a *metric*. Iff all Cauchy sequences converge for this metric, *V* is a **Hilbert space**.
(This is off-topic, but it's important in axiomatic quantum mechanics.)

★ **Complete** (Dedekind): Any non-empty subset which has an *upper bound* must have a *least upper bound*. This is used to construct $\mathbb{R}$ from *Dedekind cuts*.

★ To define "upper" and "least," we need to define *ordered sets*.

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Inner product is a "thing that behaves like a dot product."

Partial and total orders

★ A **partial order** on a set *R* is a binary relation ≤ on *R* × *R* which obeys:

1. Reflexivity $a \leq a$
2. Antisymmetry $a \leq b$ and $b \leq a \;\Rightarrow\; a = b$
3. Transitivity $a \leq b$ and $b \leq c \;\Rightarrow\; a \leq c$

★ A **total order** replaces Rule 1 with a stricter property:

1. Totality $a \leq b$ aor $b \leq a$

If ≤ is a total order, then it can be used to define ≥, <, and > in the usual way.

★ Natural numbers are the smallest totally ordered set with no upper bound.
Integers are the smallest t.o.s. with no upper or lower bound.
Rationals are the smallest *dense* t.o.s. with no upper or lower bound.
Reals are the smallest *connected* t.o.s. with no upper/lower bounds.

★ An **ordered field** is a number field with a total order which obeys:

4. $+$ respects order $a \leq b \;\Rightarrow\; a + c \leq b + c$
5. $*$ respects positivity $0 \leq a$ and $0 \leq b \;\Rightarrow\; 0 \leq a * b$

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Multidimensional reals can form a **totally ordered vector space**, but never an ordered field. (No total order on R^N is compatible with both + and *.)

Connected is defined topologically using the **order topology** defined by open intervals (a,b).

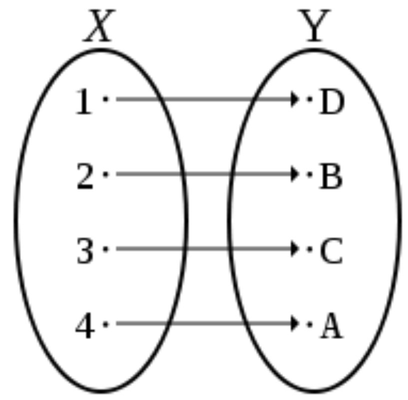
Different infinities

★ A partially ordered set is **dense** iff there is an element between any two distinct elements: $a < c \;\Rightarrow\; \exists b : a < b < c$.

Who said I was dense?

★ Example: With the usual ordering, $\mathbb{Q}$ is dense.

★ A map from a set *X* to a set *Y* is **invertible** (or **bijective**) iff every $x \in X$ is mapped to exactly one $y \in Y$ and vice versa. Two sets have the same **cardinality** iff they can be invertibly mapped to each other.



an invertible map

★ A set is **countably infinite** iff it has the cardinality of $\mathbb{N}$. The real numbers are **uncountably infinite**. A set with the same cardinality as $\mathbb{R}$ is called a **continuum**.

★ Other cardinalities exist, e.g. the **power set of the reals**, which is the set of all subsets of $\mathbb{R}$ and is denoted $2^{\mathbb{R}}$.

★ It may appear that a dense unbounded set is uncountable, but this is false! $\mathbb{Q}$ is <u>not</u> a continuum.

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Some define <u>the</u> continuum as the reals. I use it to mean any set with cardinality of the reals.

Cantor's paradise

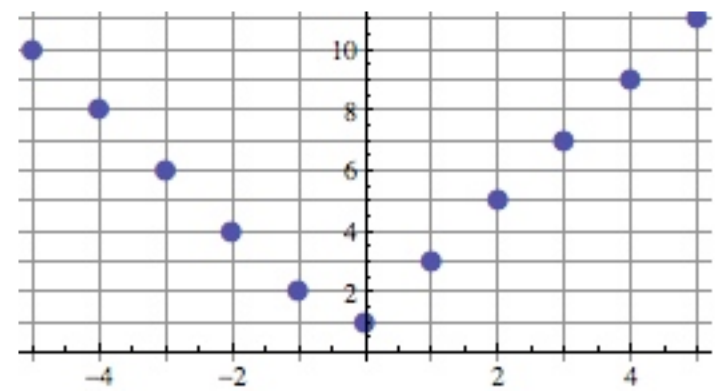
★ The cardinality of a finite set is the number of its elements. For infinite sets, the concept of "number of elements" can be confusing.

★ Example: $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, but they have the same cardinality.

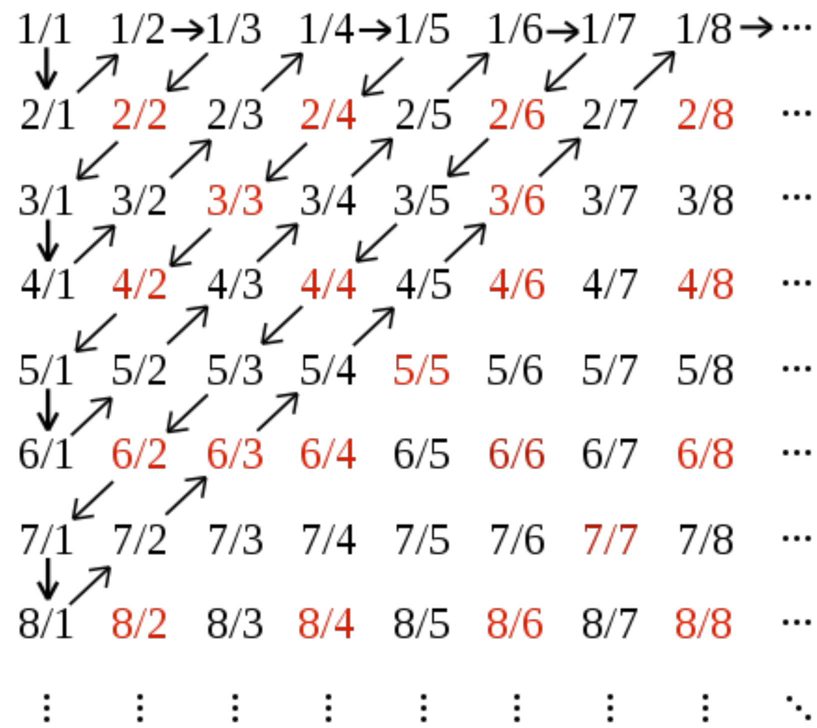
★ "Hilbert's hotel" has countably-infinite rooms and all are occupied. But if a guest walks in, we move everyone in room *n* to room *n*+1. Therefore we can always make room for one more guest.

★ The picture at bottom right shows a method for counting the positive rationals. Using this and the map *n*(*z*), we can count all of $\mathbb{Q}$.

$$n(z) = \begin{cases} -2z & \text{if } z < 0 \\ 2z+1 & \text{if } z \geq 0 \end{cases}$$



an invertible map from $\mathbb{Z}$ to $\mathbb{N}$



counting the positive rationals

bijection

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Hilbert: "No one shall expel us from the paradise that Cantor has created for us."

A **proper subset** of Z is contained in Z but does not include all of Z.

What can we do with real numbers?

Good

★ $\mathbb{R}$ is a field, so we can safely do arithmetic with $+ \; - \; * \; \div$ as long as we never divide by zero.

★ $\mathbb{R}$ is a total order, so we can rank numbers in a line.

★ $\mathbb{R}$ is complete (by all 3 definitions), so we aren't missing numbers. Sequences that should converge do converge.

This is what happens.

Don't even think about it.

Bad

★ We can't count the real numbers.

★ We can't exactly represent irrationals in positional notation (decimal or otherwise). Using Lebesgue's definition of integration, *almost all* reals are transcendental.

Banach-Tarski paradox

Ugly

★ The *Axiom of Choice* produces *non-Lebesgue-measurable* subsets of $\mathbb{R}$. Avoiding this problem requires complicated stuff like σ-*algebras.* (Dropping the AoC leads to other problems, e.g. vector spaces that have no bases.)

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Almost all means "complement of a set of measure zero."

Complex numbers

★ The **complex numbers** $\mathbb{C}$ are a number field made from ordered pairs (a,b) of real numbers. Addition and multiplication are defined like this:

$$(a, b) + (x, y) = (a + b, x + y) \qquad (a, b) * (x, y) = (ax - by, bx + ay)$$

We do <u>not</u> need to “imagine” the number $\sqrt{-1}$, but that method works as well.

★ Alternatively, we can define linear transformations on real 2D space like this:

$$\hat{\imath} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \hat{J} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

There is no real *x* such that x^2 = -1, but $\hat{J}$ is a real 2x2 matrix with $\hat{J}^2 = -\hat{\imath}$.
$\hat{J}$ represents a 90° rotation and is an *infinitesimal generator* of all 2D rotations.

★ The set of all real *linear combinations* $a\hat{\imath} + b\hat{J}$ is **field-isomorphic** to $\mathbb{C}$. That means the map $a + b\imath \leftrightarrow a\hat{\imath} + b\hat{J}$ is invertible and preserves the behavior of + and $*$ operators. These fields are said to be “the same up to isomorphism.”

★ When Hilbert said “<u>the</u> complete ordered field,” he meant that any other complete ordered field is field-isomorphic (and order-isomorphic) to the reals.

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Complex numbers can form a totally ordered **vector space**, but <u>not</u> a totally ordered **field**.
Lexicographical and *product* total orders with + are examples, but are not compatible with *.

Some other kinds of numbers

★ The *extended reals* include ±∞ and are used in Lebesgue integration. These numbers are ordered but do not form a ring. (Nor can they divide by zero.)

★ *Hyperreal numbers* are used in *nonstandard analysis*, which developed a rigorous version of Leibniz's "infinitesimals." These sets contain all the real numbers and form an ordered field, but not a complete one.

★ Hamilton's *quaternions* are a 4D division ring containing the real and complex numbers. They can be used to represent 3D rotations, which do not commute.

★ *Octonions* and *sedenions* are 8- and 16-dimensional extensions of quaternions. Like other *hypercomplex* numbers, they behave weirdly.

★ *Cardinal numbers* were designed to "count" the cardinality of infinite sets. The **continuum hypothesis** says no set has cardinality strictly between $\mathbb{N}$ and $\mathbb{R}$. The first of Hilbert's famous 23 problems for the 20th century was to verify CH. This goal is known to be impossible using standard Zermelo-Fraenkel set theory.

★ The *surreal numbers* are so confusing that I have no idea how to explain them.

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Noether taught several courses on hypercomplex numbers at Göttingen. Octonion multiplication is non-associative. Sedenions have zero divisors. Cayley-Dickson construction can be used repeatedly to build 2^N-dimensional *algebras* past the sedenions.

Irrelevant topics

★ It’s bad luck to have 13 slides. It’s also bad luck to be superstitious.

★ Some of this talk intersects with Travis Hoppe’s *Irrelevant Topics in Physics* PGSA talks, most of which are available online.

★ *The Subjugation of a Ghost* is a Zen story about a widower who believed he was haunted by the ghost of his wife. He asked a Zen master for help, who said:

"Your former wife became a ghost and knows everything you do. Whatever you do or say, whatever you give you beloved, she knows. She must be a very wise ghost. Really, you should admire such a ghost.”

"Take a large handful of soy beans and ask her exactly how many beans you hold in your hand. If she cannot tell you, you will know she is only a figment of your imagination and will trouble you no longer."

The next night, when the ghost appeared the man flattered her and told her that she knew everything.

"Indeed," replied the ghost, "and I know you went to see that Zen master today." "And since you know so much," demanded the man, "tell me how many beans I hold in this hand!"

There was no longer any ghost to answer the question.

★ Happy new year! Ask your ghosts to take a cup o’ kindness (or soy beans) for *auld lang syne*.

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For auld lang syne is old Scots. It translates (roughly) to “for old times’ sake.”