Matrix Cheat Sheet

Vectors and Linear Transformations

A **vector space** *V* is a set of things called **basis vectors** and some rules for making linear combinations of them:

a**x**+b**y** is a vector if **x**, **y** are vectors and a,b are numbers.

A **linear transformation** *L* is a map from one vector space to another that obeys the superposition principle:

L(a**x**+b**y**) = a*L***x** + b*L***y**

Every linear transformation can be represented by a matrix acting on a column vector and vice versa. This is important.

An **inner product** $\langle \mathbf{x}|\mathbf{y}\rangle$ maps two vectors to a number. The usual example is $x_1^* y_1 + x_2^* y_2 + \cdots$ but others exist. The inner product of a vector with itself defines a **norm**.

Matrix Arithmetic

To multiply two matrices *AB*, do this: $[AB]_{ij} = \sum_k A_{ik} B_{kj}$
(Note: a column vector is just a *n* x 1 matrix.)

(*AB*)**x** produces the same vector as "do *B*, then do *A* to **x**."

Matrices add component-wise, and $(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$.

To **transpose** *M*, swap its rows and columns: $[M^T]_{ij} = M_{ji}$
An **(anti) symmetric** matrix equals its (minus) transpose.

The **adjoint** of *M* is its conjugate transpose: $[M^\dagger]_{ij} = M_{ji}^*$.
Adjoint matrices obey the rule $\langle \mathbf{x}|M\mathbf{y}\rangle = \langle M^\dagger \mathbf{x}|\mathbf{y}\rangle$.

The **inverse** M^{-1} has determinant $(\det[M])^{-1}$ if $\det[M] \neq 0$.
A **singular** matrix has determinant 0 and can't be inverted.

Transposes, adjoints and inverses obey a "backwards" rule:

$$(AB)^{-1} = B^{-1}A^{-1} \quad (AB)^T = B^T A^T \quad (AB)^\dagger = B^\dagger A^\dagger$$

Unitary / Orthogonal

Unitary matrices obey $U^{-1} = U^\dagger$. Real unitary matrices are **orthogonal. *U* matrices preserve the usual inner product:** $\langle U\mathbf{x}|U\mathbf{y}\rangle = \langle \mathbf{x}|\mathbf{y}\rangle$. Each eigenvalue of *U* and the determinant of *U* must have complex magnitude 1.

The **columns of *U* form an orthonormal basis for *V*** (and so do the rows) **if and only if *U* is unitary**. Two matrices *L* and *M* are **similar** if $M = ULU^{-1}$ for some unitary *U*.

Every **rotation and/or parity transformation** between two orthonormal bases is represented by a *U* and vice versa.

Hermitian / Symmetric

Hermitian matrices are **self-adjoint**: $H^\dagger = H$. Real symmetric square matrices are Hermitian.

Eigenvalues of *H* are real (but might be degenerate!).
Eigenvectors of *H* form an orthogonal basis for *V*.
(Eigenvectors corresponding to the same eigenvalue are not unique, but it is always possible to choose orthogonal ones.)

A *real* linear combination of Hermitian matrices is Hermitian.

Eigensystems and the Spectral Theorem

A **normal** matrix *N* satisfies $NN^\dagger = N^\dagger N$. **Every normal matrix is similar to a diagonal matrix**: $N = UDU^{-1}$ where *D* is diagonal. Elements of *D* are **eigenvalues** and columns of *U* are **eigenvectors** of *N*. If *N* is Hermitian, then *U* is unitary. $\mathbf{v}_j$ is an eigenvector of *N* with eigenvalue λ_j if and only if $N\mathbf{v}_j = \lambda_j \mathbf{v}_j$. The (complex) phase of an eigenvector is arbitrary.

The **spectrum** of *N* (the set of its eigenvalues) can be found by solving $\det[N - \lambda 1] = 0$, the **characteristic polynomial** of *N*. The product of all eigenvalues of *N* is $\det[N]$ and the sum of eigenvalues is $\mathrm{tr}[N]$, the **trace** of *N* (the sum of its diagonal elements). Two similar matrices *L* and *M* have the same spectrum, determinant, and trace (but the converse is not true).

Misc. Terminology

A matrix *P* is **idempotent** if *PP* = *P*. An idempotent Hermitian matrix is a **projection**. A **positive-definite** matrix has only positive real eigenvalues. *Z* is **nilpotent** if $Z^n = 0$ for some number *n*. The **commutator** of *L* and *M* is [*L*,*M*] = *LM* - *ML* .

Matrix Exponentials

The **exponential map** of a matrix *M* is $\mathrm{EXP}[M] = 1 + M + \frac{1}{2!}M^2 + \cdots + \frac{1}{k!}M^k + \cdots$. The solution to the differential equation $\frac{d}{dt}\mathbf{x}(t) = M\mathbf{x}(t)$ is $\mathbf{x}(t) = \mathrm{EXP}[Mt] \cdot \mathbf{x}(0)$. EXP has some, but not all, of the properties of the function e^x :

in general: $(e^M)^{-1} = e^{-M} \quad (e^M)^T = e^{M^T} \quad (e^M)^\dagger = e^{M^\dagger} \quad e^{(a+b)M} = e^{aM}e^{bM} \quad \det[e^M] = e^{\mathrm{tr}[M]}$

only if *M* and *N* commute: $e^{M+N} = e^M e^N \quad e^N M e^{-N} = M$ **only** if *N* is invertible: $e^{NMN^{-1}} = Ne^M N^{-1}$.