Chapter 13
Periodic Motion

In this chapter we deal with a very special kind of motion known as oscillatory or periodic motion. It is a motion that repeats itself. Time required for each oscillation or repitition is called the period.

It is important to understand this type of motion because it occurs frequently in nature:

Motion of a mass on a spring

Oscillations of a pendulum

Motion of earth around the sun

Oscillation of atoms in a molecule or a solid

It occurs in many types of wave motion:

Sound waves: air molecules oscillate along the line of propagation of the wave

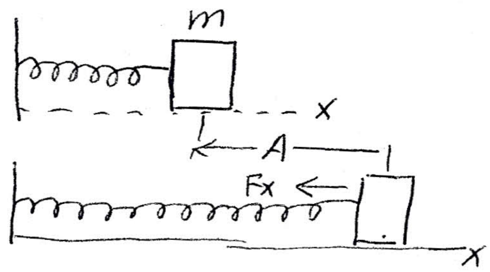
Light and Electromagnetic waves: Electric and magnetic fields oscillate perpendicular to the line of propagation of the wave.

Wave on a string or wire: The string material oscillates perpendicular to the direction of propagation of the wave.

It turns out all these periodic motion are governed by the same basic principle. When a body is displaced from its equilibrium position and is released, a force or a torque comes into play and tries to pull it back toward equilibrium. But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side and again is pulled back toward equilibrium. Picture a pendulum that swings back and forth past its straight-down position.

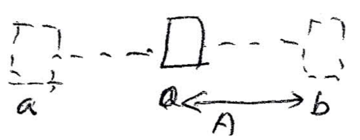
Periodic Motion of a mass attached to a spring.

To fix our ideas about periodic motion consider a mass m attached to a spring of negligible mass resting on a frictionless surface as shown in the figure. The other end



of the spring is attached to a fixed body or wall.

If we displace the mass from its equilibrium position by a A, the body will experience a restoring force F_x towards the equilibrium position and accelerates toward the equilibrium position with $a_x = \frac{F_x}{m}$. But when it comes to equilibrium position it gains kinetic energy and hence velocity in the opposite direction and so it overshoots and starts experience a restoring force in the opposite direction, Its speed decreases and it comes to a stop before starting to move in the positive x-direction again. This kind of backing and forth motion would continue for ever unless the friction of the stack would dissipate the energy of the system and ultimately stop the motion.



Amplitude, Period, Frequency and Angular Frequency

Amplitude; denoted by A is the magnitude of maximum displacement from the equilibrium position.

Complete vibration or cycle means one round trip e.g.

$a \rightarrow b \rightarrow a$ or $O \rightarrow b - O \rightarrow a \rightarrow O$.

The time period T is the time required for one complete cycle.

Number of complete cycles or vibrations per second is called the frequency, f. It has the unit of s^{-1} or Hertz.

The angular frequency, ω is defined as 2π times the frequency

$$\omega = 2\pi f$$

It represents the rate of change of an angular quantity that is measured in radians and has the unit of rad/s.

From the above definitions it is obvious that

$f = \frac{1}{T}$ or $T = \frac{1}{f}$. Hence

$\omega = 2\pi f = \frac{2\pi}{T}$.

Example: If the mass m takes 0.25 sec go from b to a. What is the period, the frequency and angular frequency?

$$T = 2 \times 0.25 = 0.5 \text{ sec}$$

$$f = \frac{1}{T} = \frac{1}{0.5 \text{sec}} = 2/s = 2 \text{ Hz}$$

$$\omega = 2\pi f = 4\pi \text{ rad/sec}.$$

Simple Harmonic Motion

In this chapter we will mostly be concerned with the simplest kind of periodic motion called the simple harmonic motion. This kind of oscillation occurs when the restoring force F_x is directly proportional to the displacement from the equilibrium $F_x = -kx$, k is the force constant of the spring ($\frac{N}{m}$)

The minus sign is inserted to make sure that the equation shows the correct direction of the force.

Since force $F_x = ma_x$, we get the equation of simple harmonic osillation as

$$ma_x = -kx \Rightarrow a_x \equiv \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Defining $\omega^2 = \frac{k}{m}$, our equation of motion becomes

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

It ~~our~~ turns out that the angular frequency ω that we defined for a periodic motion becomes exactly equal to the $\omega = \sqrt{\frac{k}{m}}$ for the simple harmonic motion. This can be shown by comparing the simple harmonic motion with uniform circular motion of a particle in a circle of radius A.

It turns out that the general solution of this second order differential equation is a sine or a cosine function of time i.e. we can write

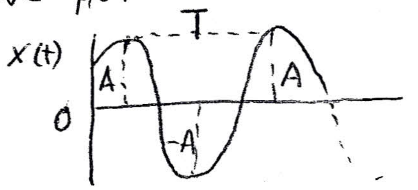
$$x(t) = A\cos(\omega t + \phi)$$

[It can be easily checked that this function satisfies the above differential equation: $\frac{dx}{dt} = -A\omega \sin(\omega t + \phi)$

$$\frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t+\phi) = -\omega^2 x$$

so substitution shows that the DE is satisfied]

Thus in simple harmonic motion the position is a periodic sinusoidal function of time. A, ω and φ are constants. If we plot $x(t)$ as a function of time we would get



The magnitude of the maximum displacement A is the amplitude.

$(\omega t + \phi)$ is called the phase of motion and φ is the phase constant, A and φ can be found from the initial conditions, i.e. initial displacement x_0 and initial speed v_0

From the figure we see that

$$x(t+T) = x(t)$$

$$\Rightarrow \cos[\omega(t+T)+\phi] = \cos(\omega t+\phi) = \cos[\omega t + 2\pi + \phi]$$

$$\Rightarrow \omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$$

Thus $f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ or ~~$\omega = 2\pi f$~~ $T = 2\pi\sqrt{\frac{m}{k}}$

We can also obtain speed and acceleration of SHM

$$v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi)$$

$$|v_{max}| = \omega A, \quad |a_{max}| = \omega^2 A$$

To find A and ϕ from the initial condition, we use

$$x(0) = x_0 = A\cos\phi$$
$$v_x(0) = v_0 = -A\omega\sin\phi \quad \Rightarrow \quad A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

and $\phi = \cos^{-1}\frac{x_0}{A}$ or $\sin^{-1}\left(-\frac{v_0}{\omega A}\right)$ both of these should be satisfied.

or $\frac{v_0}{\omega x_0} = -\tan\phi \Rightarrow \phi = \arctan\left(-\frac{v_0}{\omega x_0}\right)$

Example 200g block connected to a light spring of spring constant $k = 5\,\frac{N}{m}$. Block is released from an initial displacement $x_0 = 5\,cm$ and initial velocity $v_0 = -1\,m/s$.

a) Calculate T : $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5\,N/m}{0.2\,kg}} = 5\,rad/s$

$$T = \frac{2\pi}{\omega} = 0.26\,s$$

b) Calculate amplitude and ϕ

$$\tan\phi = -\frac{v_0}{\omega x_0} = \frac{0.100}{(5)(0.05)} = 0.40$$

$$\phi = 0.12\pi$$

$$A = \frac{x_0}{\cos\phi} = \frac{0.05}{\cos(0.12\pi)} = 0.0539\,m$$

c) $v_{max} = \omega A = (5)(5.30 \times 10^{-2}) = 0.269\,m/s$

d) $a_{max} = \omega^2 A = (5)^2(5.39 \times 10^{-2}) = 1.35\,m/s^2$

e) $x(t) = (0.0539\,m)\cos(5t + 0.12\pi)$
$v_x(t) = -(0.269\,m/s)\sin(5t + 0.12\pi)$
$a_x(t) = -(1.35\,m/s^2)\cos(5t + 0.12\pi)$