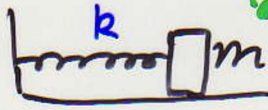
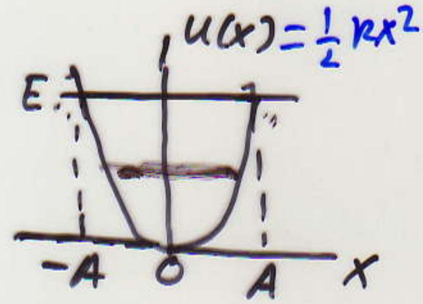
The Harmonic Oscillator



$\omega = \sqrt{\frac{k}{m}}$

Classically harmonic oscillator is a mass attached to spring of spring constant. Restoring force $F = -kx$ (k not wave number)

Potential energy $U = \frac{1}{2}kx^2$
$= \frac{1}{2}m\omega^2 x^2$



Newtonian Picture

1. Particle more likely to be found at "turning points of motion" where it is slowest.
2. Particle can have <u>any</u> energy.

Quantum Picture

Schroedinger eqn satisfied by a particle in a harmonic oscillator potential

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \underbrace{U(x)}_{\frac{1}{2}kx^2}\psi(x) = E\psi(x)$$

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}\left(\frac{1}{2}kx^2 - E\right)\psi(x)$$

The solution of this eqn is not as easy as the case of a particle in a box. It has been solved, however, with boundary conditions

that the wave function goes to zero at $x = -\infty$ and ∞.
The solutions involve Hermite polynomials consisting of an exponential function multiplied by a polynomial.

State with lowest energy has the form

$$\psi_0(x) = C\, e^{-\sqrt{mk}\, x^2/2\hbar}$$

and lowest energy

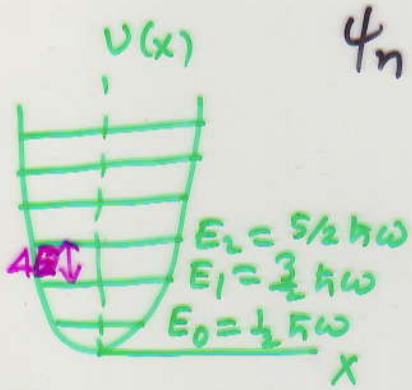
$$E_0 = \frac{\hbar\omega}{2}, \quad \neq 0, \text{ zero point energy.}$$

Other allowed energy eigen values and eigenfunctions are

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2 \cdots$$

ψ_n = complicated function involving Hermite polynomials.

$\Delta E = \hbar\omega$

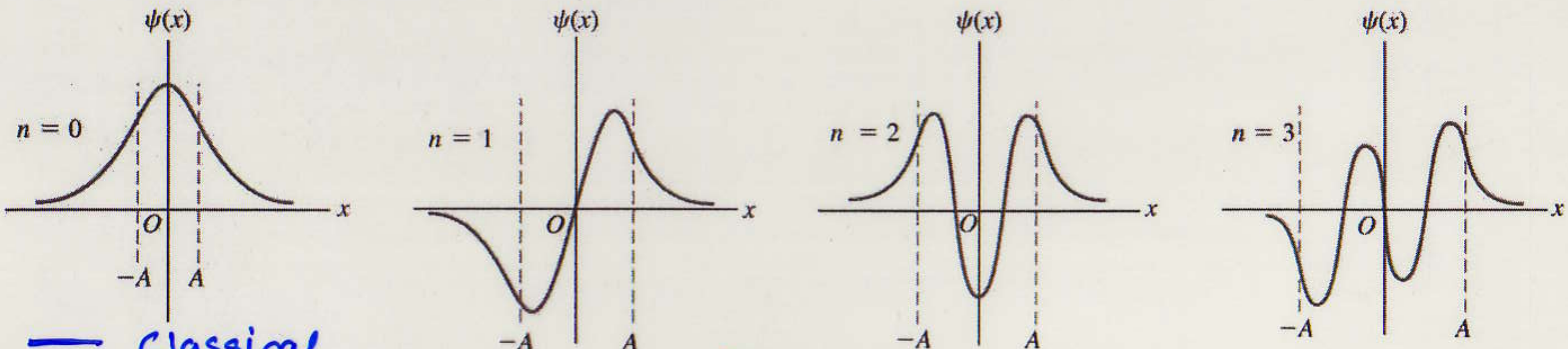


Equally spaced energy levels.

It is interesting to plot the wave functions and the corresponding quantum probability distribution along with the classical probability distribution for different allowed energy states, as shown on the next page.

A comparison of Newtonian and quantum oscillators

- ~~As I said on the previous slide, "they're analogous." I didn't say "they're the same."~~
- Consider Figure 40.19 below.
- Consider Figure 40.20 below.



Classical
Quantum

Fig. 40.19

tail in Classically forbidden zone

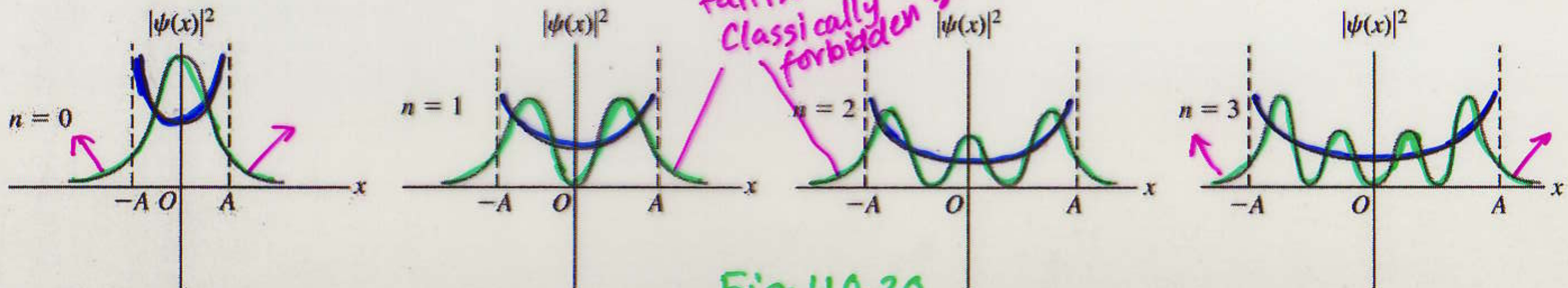
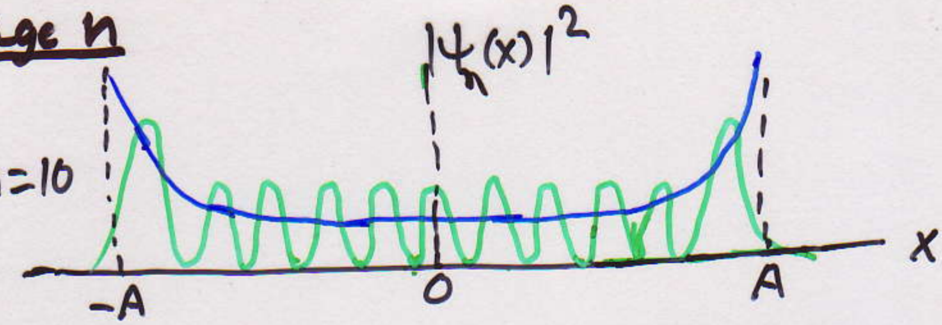


Fig. 40.20

Large n



Note that $|\psi_n|^2$ for large $n \longrightarrow$ classical picture.
i.e. Quantum picture approaches classical/Newtonian picture for large quantum numbers n.

It can be easily seen that energy E of quantum mechanical harmonic oscillator cannot be equal to zero, because that would violate the uncertainty principle. Minimum energy must be $\frac{\hbar\omega}{2}$ to conform to uncertainty principle. To see this consider

$$E = \frac{1}{2}kA^2 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{\frac{k}{m}} \Rightarrow A = \sqrt{\frac{\hbar}{\sqrt{km}}}$$

A corresponds to maximum uncertainty in $x = \Delta x$

Now $$p_{max} = m v_{max} = m\sqrt{\frac{k}{m}}\,A = m\sqrt{\frac{k}{m}}\sqrt{\frac{\hbar}{\sqrt{km}}}$$
$$= \sqrt{\hbar\sqrt{km}}$$

p_{max} corresponds to Δp_x.

Thus $$\Delta x \Delta p_x = \sqrt{\frac{\hbar}{\sqrt{km}}}\sqrt{\hbar\sqrt{km}} = \hbar$$

Thus the minimum energy state satisfies uncertainty principle.

PRACTICE FINAL

1. Please note that each part of this problem is on a different topic.

(a) In the Rutherford scattering experiment a particular α particle makes a direct head-on collision with a gold nucleus ($^{197}_{79}Au$) and scatters backward at 180°. If the distance of closest approach of the α particle is found to be $d = 4.29\text{x}10^{-14}$ m, what was the initial kinetic energy of the α particle?

(b) A 5.1 eV α particle is incident on a rectangular barrier of height 6.8 eV. What is the approximate probability that the α particle will tunnel though the barrier if the barrier width is 0.75 nm?

(c) A black, totally absorbing piece of cardboard of area A= 2.0 cm^2 intercepts light with an intensity of 10 W/m^2 from a camera strobe light. What radiation pressure is produced on the cardboard by the light?

(d) The total energy of a mass oscillating on a spring is 6 J. What are the kinetic and potential energies of this mass when its displacement is half of its amplitude?

(20 points)

1. (a) $U_f = k_e \frac{q_{Au} q_\alpha}{d} = K_i$, $k_e = \frac{1}{4\pi\epsilon_0}$

$$K_i = (8.99\times10^9)\frac{(2)(79)e^2}{4.29\times10^{-14}} = \frac{(8.99\times10^9)(2)(79)(1.6\times10^{-19})^2}{(4.29\times10^{-14})} = 8.48\times10^{-15}\text{J}$$

$= 5.29$ MeV

(b) $T \approx e^{-2kL}$ $\quad G \approx 1$

$$k = \frac{\sqrt{2m(U-E)}}{\hbar} = \frac{\sqrt{2(6.6447\times10^{-27})(6.8-5.1)(1.6\times10^{-19})}}{1.055\times10^{-34}}$$

$$= \frac{6.01\times10^{-23}}{1.055\times10^{-34}} = 5.69\times10^{11}$$

$$2kL = 2(5.69\times10^{11})(0.75\times10^{-9}) = 8.535\times10^2$$

$$T = e^{-853.5} \longrightarrow 0$$

(c) $P = \frac{I}{c} = \frac{10\ \text{W/m}^2}{3\times10^8\ \text{m/s}} = 3.33\times10^{-8}\ \frac{\text{J}}{\text{m}^3} = 3.33\times10^{-8}\ \frac{\text{N}}{\text{m}^2}$

(d)

$$E = \tfrac{1}{2}kA^2 = 6\text{J}$$

$$U = \tfrac{1}{2}k\left(\tfrac{A}{2}\right)^2 = \tfrac{1}{2}kA^2/4 = \frac{E}{4} = \frac{6}{4} = 1.5\text{J}$$

$$K = \tfrac{1}{2}kA^2 - \tfrac{1}{8}kA^2 = \tfrac{1}{2}kA^2\left(1-\tfrac{1}{4}\right) = \tfrac{3}{4}\left(\tfrac{1}{2}kA^2\right) = \frac{3}{\cancel{4}_2}\times\cancel{6}^3\text{J}$$

$$= \frac{9}{2} = 4.5\text{J}.$$

2. Please note that each part of this problem is on a different topic.

(a) (i) The work function for cesium is 1.9 eV. Find the threshold (maximum) wavelength of the incident photons for producing the photoelectric effect. (ii) If the intensity of the incident photons is doubled, what would be the threshold wavelength?

(b) The energy levels for the electrons of hydrogen atom are given by

$$E_n = \frac{-13.6eV}{n^2}, \text{ where } n=1, 2, 3, \ldots$$

The Balmer series corresponds to transition to the state n=2. The most prominent line in this series corresponds to red light. The other lines correspond to colors in the blue to ultraviolet range. What is the wavelength of the red line?

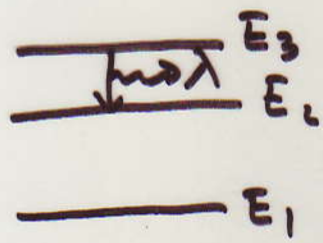
(c) Electrons in an electron microscope are accelerated through a potential difference $\Delta V = 10^4 V$. What is the de Broglie wavelength of these electrons?

(d) The energy of a certain nuclear state can be measured with an uncertainty of 1 eV. What is the minimum lifetime of this state?

2 a) (i) $\frac{hc}{\lambda_t} = \phi \Rightarrow \lambda_c = \frac{hc}{\phi} = \frac{1240 \text{ eV-nm}}{1.9 \text{ eV}} = \boxed{653 \text{ nm}}$

(ii) Will remain the same

b) $hf = E_3 - E_2 \Rightarrow \frac{hc}{\lambda} = E_3 - E_2 \Rightarrow \lambda = \frac{hc}{E_3 - E_2}$



$E_3 - E_2 = -13.6\left[\frac{1}{3^2} - \frac{1}{2^2}\right] = \frac{(-13.6)(-5)}{(9)(4)} = 1.89 \text{ eV}$

$\Rightarrow \lambda = \frac{1240 \text{ eV-nm}}{1.89 \text{ eV}} = \boxed{656 \text{ nm}}$

c) $\lambda = \frac{h}{p} = \frac{h}{mv}, \quad \frac{1}{2}mv^2 = |e|\Delta V \Rightarrow v = \sqrt{\frac{2|e|\Delta V}{m}}$

$\lambda = \frac{h}{m\sqrt{\frac{2|e|\Delta V}{m}}} = \frac{h}{\sqrt{2|e|\Delta V m}} = \frac{6.63 \times 10^{-34}}{\sqrt{2(1.6\times10^{-19})(10^4)(9.11\times10^{-31})}}$

$= 0.0123 \text{ nm}$

d) $\tau = \Delta t \geq \frac{\hbar}{\cancel{2}(\Delta E)} = \frac{1.05 \times 10^{-34} \text{ J.s}}{(\cancel{2})(1 \text{ eV})(1.6\times10^{-19} \frac{J}{eV})} = \boxed{3.28 \times 10^{-16} \text{ s}} \times 2$

3. An electron is confined to a one-dimensional infinitely deep potential well of width L=100 pm.
(a) What is the ground state energy of this electron, i.e. what is its energy at state n=1?
(b) What is the probability that the electron can be detected within an interval dx=0.01L at the midpoint (x=L/2) of the potential well?
(c) If the electron makes a transition from the ground state (n=1) to the second excited sate (n=3) by absorbing light, what must be the wavelength of this light?
(d) Once the electron has been excited to the second excited state, what wavelengths of light can it emit by de-excitation? (20 points)

3. (a) $E_1 = \frac{h^2}{8mL^2} = \frac{(6.626\times10^{-34})^2}{8(9.11\times10^{-31})(1\times10^{-10})^2} = \boxed{6.03\times10^{-18}\,J}$

$= \frac{6.03\times10^{-18}\,J}{1.6\times10^{-19}\,J/eV} = \boxed{37.6\,eV}$

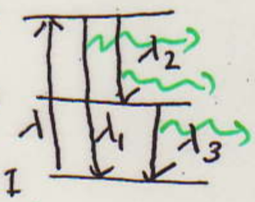
(b) $P(x)\,dx = |\psi_1(x=\frac{L}{2})|^2(0.01L) = \frac{2}{L}\sin^2(\frac{\pi}{L}\frac{L}{2})(0.01L) = \frac{2}{L}(0.01L)$

$= 0.02$

(c) $\frac{hc}{\lambda} = E_3 - E_1 = 3^2E_1 - E_1 = 8E_1$

$\lambda = \frac{hc}{8E_1} = \frac{1240\text{ eV-nm}}{8(37.6\text{ eV})} = \boxed{4.12\text{ nm}}$

(d) $\lambda_1 = \lambda = 4.12\,nm$



$\frac{hc}{\lambda_2} = E_3 - E_2 = 9E_1 - 4E_1 = 5E_1$

$\lambda_2 = \cancel{5E_1}\,\frac{hc}{5E_1} = \frac{1240\text{ eV-nm}}{5(37.6\text{ eV})} = \boxed{6.6\text{ nm}}$

$\frac{hc}{\lambda_3} = E_2 - E_1 = 4E_1 - E_1 = 3E_1, \Rightarrow \lambda_3 = \frac{hc}{3E_1} = \frac{1240}{3(37.6)} = \boxed{11.0\text{ nm}}$

4. The proper mean lifetime of π mesons is 2.6×10^{-8} sec. A beam of such particles is moving with a speed 0.9c with respect to the earth.
(a) What is their mean lifetime as measured in the laboratory on the earth?
(b) How far would they travel before they decay as measured in the laboratory on the earth?
(c) How far would they travel before they decay as measured by someone traveling with the π mesons?
(d) In the reference of the π mesons, how far does the laboratory travel in a typical lifetime of 2.6×10^{-8} sec? (20 points)

4

a) $\tau = \gamma \tau_0 \qquad \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-(0.9)^2}} = 2.29$

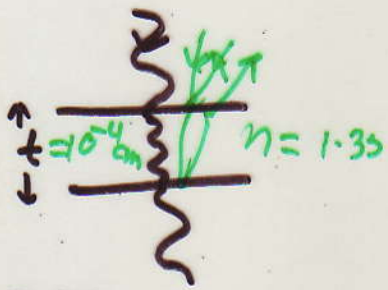
$\Rightarrow \tau = (2.29)(2.6 \times 10^{-8} s) = \cancel{\boxed{5.95 \text{ sec}}}\ \boxed{5.95 \times 10^{-8} \text{ sec}}$

(b) $x = u\tau = (0.9c)(5.95 \times 10^{-8} \text{ sec}) = (0.9)(3 \times 10^8)(5.95 \times 10^{-8})$
$= \boxed{16.1 \text{ m}}$

(c) In this frame, π mesons are not moving $\Rightarrow x = 0$

(d) $x' = u\tau_0 = (0.9c)(2.6 \times 10^{-8} \text{sec}) = (0.9)(3 \times 10^8)(2.6 \times 10^{8}) = \boxed{7.02 \text{ m}}$

5. Light of wavelength 500 nm is incident normally on thin film of water 10^{-4} cm thick. The refractive index of water is 1.33.
(a) What is the wavelength of the light in the water?
(b) How many wavelengths are contained in the distance 2t inside the film, where t is the thickness of the film?
(c) What is the total phase difference between the wave reflected from the top of the film and the one reflected from the bottom of the film?
(d) What would be the minimum thickness of the film such that the two reflected waves (one from the top and the other from the bottom of the film) may interfere constructively? (20 points)



5.

$t = 1\times10^{-4}\,cm = 1\times10^{-6}\,m$

$n = 1.33$

a) $\lambda_n = \frac{\lambda_v}{n} = \frac{500}{1.33} = 375.94\,nm$

b) $N\lambda_n = 2t \Rightarrow N = \frac{2t}{\lambda_n} = \frac{2(10^{-6})}{(375.94)(10^{-9})} = 5.32$

c) Total phase difference

$\phi = k\delta - \pi$

$$\phi = \underbrace{\frac{2\pi}{\lambda_n}(2t)}_{\text{path diff}} - \pi \leftarrow \text{phase change due to reflection at denser medium}$$

$$= \frac{2\pi}{\lambda_v}(2nt) - \pi = \left(\frac{4nt}{\lambda} - 1\right)\pi$$

$$= \left(\frac{4(1.33)(10^{-6})}{5\times10^{-7}} - 1\right)\pi = 9.64\pi = 30.28\ rad$$

d) $\frac{4nt}{\lambda_v} - 1 = 0 \Rightarrow t = \frac{\lambda_v}{4n} = \frac{500nm}{4(1.33)} = 94nm$

6. A rope, under the tension of 200 N and fixed at both ends, oscillates in a second harmonic standing wave pattern as shown in the figure below. The displacement of the rope is given by

$$y(x,t) = [0.10m(\sin \pi x/2)]\sin 12\pi t,$$

where x=0 at one end of the rope, x is in meters and t is in seconds. What are

(a) the values of the wave number k and the angular frequency ω of the waves forming the standing wave on the rope,
(b) the speed of the waves on the rope,
(c) the length of the rope, and
(d) the mass of the rope?

(30 points)

a) $k = \boxed{\frac{\pi}{2}/m}$, $\omega = \boxed{12\pi/s}$ since $y = [2A \overset{\sin}{\cos} kx] \sin\omega t$

b) $v = \frac{\omega}{k} = \frac{12\pi}{\pi/2} = \boxed{24\ m/s}$

c) $k = \frac{2\pi}{\lambda}$ or $\frac{2\pi}{\lambda} = \frac{\pi}{2} \Rightarrow \lambda = 4m$

$L = \lambda$ (see figure) $\Rightarrow L = \boxed{4m}$

d) $v = \sqrt{\frac{T}{\mu}} \Rightarrow \mu = \frac{T}{v^2} \Rightarrow m = \mu l = l\left(\frac{T}{v^2}\right)$

$= 4\left(\frac{200}{24^2}\right) = \boxed{1.39\ kg}$