

Lie Groups, Physics and Geometry

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Many years ago I wrote the book *Lie Groups, Lie Algebras, and Some of Their Applications* (NY: Wiley, 1974). That was a big book: long and difficult. Over the course of the years I realized that more than 90% of the most useful material in that book could be presented in less than 10% of the space. This realization was accompanied by a promise that some day I would do just that — rewrite and shrink the book to emphasize the most useful aspects in a way that was easy for students to acquire and to assimilate. The present work is the fruit of this promise.

In carrying out the revision I've created a sandwich. Lie group theory has its intellectual underpinnings in Galois theory. In fact, the original purpose of what we now call Lie group theory was to use continuous groups to solve differential (continuous) equations in the spirit that finite groups had been used to solve algebraic (finite) equations. It is rare that a book dedicated to Lie groups begins with Galois groups and includes a chapter dedicated to the applications of Lie group theory to solving differential equations. This book does just that. The first chapter describes Galois theory, and the last chapter shows how to use Lie theory to solve some ordinary differential equations. The fourteen intermediate chapters describe many of the most important aspects of Lie group theory and provide applications of this beautiful subject to several important areas of physics and geometry.

Over the years I have profitted from the interaction with many students through comments, criticism, and suggestions for new material or different approaches to old. Three students who have contributed enormously during the past few years are Dr. Jairzinho Ramos-Medina, who worked with me on Chapter 15 (Maxwell's Equations), and Daniel J. Cross and Timothy Jones, who aided this computer illiterate with much moral and ebite ether support. Finally, I thank my beautiful wife Claire for her gracious patience and understanding throughout this long creation process.

Contents

1	Introduction	<i>page</i> 1
1.1	The Program of Lie	1
1.2	A Result of Galois	3
1.3	Group Theory Background	4
1.4	Approach to Solving Polynomial Equations	9
1.5	Solution of the Quadratic Equation	10
1.6	Solution of the Cubic Equation	12
1.7	Solution of the Quartic Equation	15
1.8	The Quintic Cannot be Solved	18
1.9	Example	19
1.10	Conclusion	22
1.11	Problems	23
2	Lie Groups	25
2.1	Algebraic Properties	25
2.2	Topological Properties	27
2.3	Unification of Algebra and Topology	29
2.4	Unexpected Simplification	31
2.5	Conclusion	31
2.6	Problems	32
3	Matrix Groups	37
3.1	Preliminaries	37
3.2	No Constraints	39
3.3	Linear Constraints	39
3.4	Bilinear and Quadratic Constraints	42
3.5	Multilinear Constraints	46
3.6	Intersections of Groups	46
3.7	Embedded Groups	47

3.8	Modular Groups	48
3.9	Conclusion	50
3.10	Problems	50
4	Lie Algebras	61
4.1	Why Bother?	61
4.2	How to Linearize a Lie Group	63
4.3	Inversion of the Linearization Map: EXP	64
4.4	Properties of a Lie Algebra	66
4.5	Structure Constants	68
4.6	Regular Representation	69
4.7	Structure of a Lie Algebra	70
4.8	Inner Product	71
4.9	Invariant Metric and Measure on a Lie Group	74
4.10	Conclusion	76
4.11	Problems	76
5	Matrix Algebras	82
5.1	Preliminaries	82
5.2	No Constraints	83
5.3	Linear Constraints	83
5.4	Bilinear and Quadratic Constraints	86
5.5	Multilinear Constraints	89
5.6	Intersections of Groups	89
5.7	Algebras of Embedded Groups	90
5.8	Modular Groups	91
5.9	Basis Vectors	91
5.10	Conclusion	93
5.11	Problems	93
6	Operator Algebras	98
6.1	Boson Operator Algebras	98
6.2	Fermion Operator Algebras	99
6.3	First Order Differential Operator Algebras	100
6.4	Conclusion	103
6.5	Problems	104
7	EXPonentiation	110
7.1	Preliminaries	110
7.2	The Covering Problem	111
7.3	The Isomorphism Problem and the Covering Group	116
7.4	The Parameterization Problem and BCH Formulas	121
7.5	EXPonentials and Physics	127

	7.5.1	Dynamics	127
	7.5.2	Equilibrium Thermodynamics	129
	7.6	Conclusion	132
	7.7	Problems	133
8		Structure Theory for Lie Algebras	145
	8.1	Regular Representation	145
	8.2	Some Standard Forms for the Regular Representation	146
	8.3	What These Forms Mean	149
	8.4	How to Make This Decomposition	152
	8.5	An Example	153
	8.6	Conclusion	154
	8.7	Problems	154
9		Structure Theory for Simple Lie Algebras	157
	9.1	Objectives of This Program	157
	9.2	Eigenoperator Decomposition – Secular Equation	158
	9.3	Rank	161
	9.4	Invariant Operators	161
	9.5	Regular Elements	164
	9.6	Semisimple Lie algebras	166
	9.6.1	Rank	166
	9.6.2	Properties of Roots	166
	9.6.3	Structure Constants	168
	9.6.4	Root Reflections	169
	9.7	Canonical Commutation Relations	169
	9.8	Conclusion	171
	9.9	Problems	173
10		Root Spaces and Dynkin Diagrams	179
	10.1	Properties of Roots	179
	10.2	Root Space Diagrams	181
	10.3	Dynkin Diagrams	185
	10.4	Conclusion	189
	10.5	Problems	191
11		Real Forms	194
	11.1	Preliminaries	194
	11.2	Compact and Least Compact Real Forms	197
	11.3	Cartan’s Procedure for Constructing Real Forms	199
	11.4	Real Forms of Simple Matrix Lie Algebras	200
	11.4.1	Block Matrix Decomposition	201
	11.4.2	Subfield Restriction	201

11.4.3	Field Embeddings	204
11.5	Results	204
11.6	Conclusion	205
11.7	Problems	206
12	Riemannian Symmetric Spaces	213
12.1	Brief Review	213
12.2	Globally Symmetric Spaces	215
12.3	Rank	216
12.4	Riemannian Symmetric Spaces	217
12.5	Metric and Measure	218
12.6	Applications and Examples	219
12.7	Pseudo Riemannian Symmetric Spaces	222
12.8	Conclusion	223
12.9	Problems	224
13	Contraction	232
13.1	Preliminaries	233
13.2	Inönü–Wigner Contractions	233
13.3	Simple Examples of Inönü–Wigner Contractions	234
13.3.1	The Contraction $SO(3) \rightarrow ISO(2)$	234
13.3.2	The Contraction $SO(4) \rightarrow ISO(3)$	235
13.3.3	The Contraction $SO(4, 1) \rightarrow ISO(3, 1)$	237
13.4	The Contraction $U(2) \rightarrow H_4$	239
13.4.1	Contraction of the Algebra	239
13.4.2	Contraction of the Casimir Operators	240
13.4.3	Contraction of the Parameter Space	240
13.4.4	Contraction of Representations	241
13.4.5	Contraction of Basis States	241
13.4.6	Contraction of Matrix Elements	242
13.4.7	Contraction of BCH Formulas	242
13.4.8	Contraction of Special Functions	243
13.5	Conclusion	244
13.6	Problems	245
14	Hydrogenic Atoms	250
14.1	Introduction	251
14.2	Two Important Principals of Physics	252
14.3	The Wave Equations	253
14.4	Quantization Conditions	254
14.5	Geometric Symmetry $SO(3)$	257
14.6	Dynamical Symmetry $SO(4)$	261

14.7	Relation With Dynamics in Four Dimensions	264
14.8	DeSitter Symmetry $SO(4, 1)$	266
14.9	Conformal Symmetry $SO(4, 2)$	270
14.9.1	Schwinger Representation	270
14.9.2	Dynamical Mappings	271
14.9.3	Lie Algebra of Physical Operators	274
14.10	Spin Angular Momentum	275
14.11	Spectrum Generating Group	277
14.11.1	Bound States	278
14.11.2	Scattering States	279
14.11.3	Quantum Defect	280
14.12	Conclusion	281
14.13	Problems	282
15	Maxwell's Equations	293
15.1	Introduction	294
15.2	Review of the Inhomogeneous Lorentz Group	295
15.2.1	Homogeneous Lorentz Group	295
15.2.2	Inhomogeneous Lorentz Group	296
15.3	Subgroups and Their Representations	296
15.3.1	Translations $\{I, a\}$	297
15.3.2	Homogeneous Lorentz Transformations	297
15.3.3	Representations of $SO(3, 1)$	298
15.4	Representations of the Poincaré Group	299
15.4.1	Manifestly Covariant Representations	299
15.4.2	Unitary Irreducible Representations	300
15.5	Transformation Properties	305
15.6	Maxwell's Equations	308
15.7	Conclusion	309
15.8	Problems	310
16	Lie Groups and Differential Equations	320
16.1	The Simplest Case	322
16.2	First Order Equations	323
16.2.1	One Parameter Group	323
16.2.2	First Prolongation	323
16.2.3	Determining Equation	324
16.2.4	New Coordinates	325
16.2.5	Surface and Constraint Equations	326
16.2.6	Solution in New Coordinates	327
16.2.7	Solution in Original Coordinates	327

16.3	An Example	327
16.4	Additional Insights	332
16.4.1	Other Equations, Same Symmetry	332
16.4.2	Higher Degree Equations	333
16.4.3	Other Symmetries	333
16.4.4	Second Order Equations	333
16.4.5	Reduction of Order	335
16.4.6	Higher Order Equations	336
16.4.7	Partial Differential Equations: Laplace's Equation	337
16.4.8	Partial Differential Equations: Heat Equation	338
16.4.9	Closing Remarks	338
16.5	Conclusion	339
16.6	Problems	341
	<i>Bibliography</i>	347
	<i>Index</i>	351

