2
Lie Groups

Contents

Lie groups are beautiful, important, and useful because they have one foot in each of the two great divisions of mathematics — algebra and geometry. Their algebraic properties derive from the group axioms. Their geometric properties derive from the identification of group operations with points in a topological space. The rigidity of their structure comes from the continuity requirements of the group composition and inversion maps. In this chapter we present the axioms that define a Lie group.

2.1 Algebraic Properties

The algebraic properties of a Lie group originate in the axioms for a group.

Definition: A set $g_i, g_j, g_k, \ldots$ (called *group elements* or *group operations*) together with a combinatorial operation $\circ$ (called *group multiplication*) form a group G if the following axioms are satisfied:

(i) **Closure:** If $g_i \in G$, $g_j \in G$, then $g_i \circ g_j \in G$.

(ii) **Associativity:** $g_i \in G$, $g_j \in G, g_k \in G$, then

$$(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$$

(iii) **Identity:** There is an operator e (the *identity operation*) with the property that for every group operation $g_i \in G$

$$g_i \circ e = g_i = e \circ g_i$$

(iv) **Inverse:** Every group operation g_i has an inverse (called g_i^{-1}) with the property

$$g_i \circ g_i^{-1} = e = g_i^{-1} \circ g_i$$

Example: We consider the set of real 2×2 matrices $SL(2;R)$:

$$A = \left[\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right] \qquad \det(A) = \alpha\delta - \beta\gamma = +1 \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers. This set forms a group under matrix multiplication. This is verified by checking that the group axioms are satisfied.

(i) **Closure:** If A and B are real 2×2 matrices, and $A \circ B = C$ (where $\circ$ now represents matrix multiplication), then C is a real 2×2 matrix. If $\det(A) = +1$ and $\det(B) = +1$, then $\det(C) = \det(A)\det(B) = +1$.

(ii) **Associativity:** $(A \circ B) \circ C$ and $A \circ (B \circ C)$ are given explicitly by

$$\begin{array}{ccc} \sum_k (\sum_j A_{ij} B_{jk}) C_{kl} & \stackrel{?}{=} & \sum_j A_{ij} (\sum_k B_{jk} C_{kl}) \\ \sum_k \sum_j A_{ij} B_{jk} C_{kl} & \stackrel{\text{ok}}{=} & \sum_j \sum_k A_{ij} B_{jk} C_{kl} \end{array} \tag{2.2}$$

(iii) **Identity:** The unit matrix is the identity

$$e \longrightarrow I_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

(iv) **Inverse:** The unique matrix inverse of A is

$$\left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \rightarrow \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \left[\begin{array}{cc} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{array} \right]$$

2.2 Topological Properties

The geometric structure of a Lie group comes from the identification of each element in the group with a point in some topological space: $g_i \rightarrow g(x)$. In other words, the index i depends on one or more continuous real variables.

The topological space that parameterizes the elements in a Lie group is a manifold. A *manifold* is a space that looks Euclidean on a small scale everywhere. For example, every point on the surface of a unit sphere $S^2 \subset R^3$: $x^2 + y^2 + z^2 = 1$, has a neighborhood that looks, over small distances, like a piece of the plane R^2 (cf. Fig. 2.1). Locally, the two spaces S^2 and R^2 are topologically equivalent but globally they are different (Columbus).

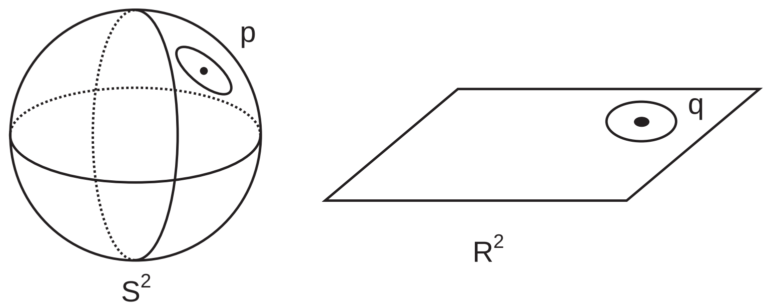

Fig. 2.1. Every point p on a sphere S^2 is surrounded by an open neighborhood that is indistinguishable from an open neighborhood of any point in the plane R^2. Locally the two spaces are indistinguishable. Globally they are distinguishable.

Definition: An n-dimensional differentiable manifold M^n consists of

(i) A topological space T. This includes a collection of open sets U_α (a topology) that cover T: $\cup_\alpha U_\alpha = T$.

(ii) A collection of charts ϕ_α, with $\phi_\alpha(U_\alpha) = V_\alpha \subset R^n$. Each ϕ_α is a homeomorphism of U_α to V_α.

(iii) Smoothness conditions: The homeomorphisms $\phi_\alpha \circ \phi_\beta^{-1}$: $\phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$ of open sets in R^n to open sets in R^n are 1:1, invertible, and differentiable.

Remarks: The charts ϕ_α allow construction of coordinate systems on the open sets U_α. It is often not possible to find a single coordinate system on the entire manifold, as the example of the sphere in Fig.

2.1 shows. Since the "transition functions" $\phi_\alpha \circ \phi_\beta^{-1}$ map $R^n \rightarrow R^n$, all the definitions of elementary multivariable calculus are applicable to them. For example, the adjective "differentiable" can be replaced by other adjectives (C^k, smooth, analytic, ...) in the definition above.

Example: Real 2×2 matrices are identified by four real variables. The unimodular condition $\det(A) = +1$ places one constraint on these four real variables. Therefore every group element in $SL(2;R)$ is determined by a point in some real three-dimensional space. One possible parameterization is

$$(x_1, x_2, x_3) \longrightarrow \left[\begin{array}{cc} x_1 & x_2 \\ x_3 & \dfrac{1+x_2x_3}{x_1} \end{array}\right] \qquad\qquad x_1 \neq 0 \qquad (2.3)$$

Parameterization of the operations in a group by real numbers is a nontrivial problem, as is clear when one asks: "What happens as $x_1 \rightarrow 0$?" We will consider this question in Chapter 5.

The manifold that parameterizes the group $SL(2;R)$ is the direct product manifold R^2 (plane) $\times S^1$ (circle) (cf. Fig. 2.2). This is not at all obvious, but will become clear when we discuss the infinitesimal properties of Lie groups in Chapter 4.

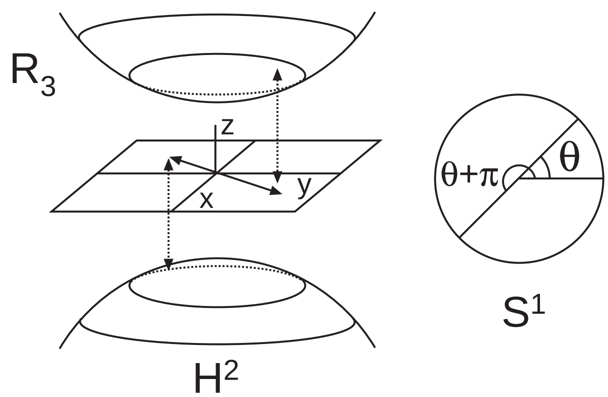

Fig. 2.2. Every matrix in $SL(2;R)$ can be written as the product of a symmetric matrix and a rotation matrix, both unimodular. The symmetric matrix is parameterized by a 2-dimensional manifold, the 2-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$. The rotation matrix is parameterized by a point on a circle. The parameterization manifold, $H^2 \times S^1$, is three dimensional.

The dimension of the manifold that parameterizes a Lie group is the dimension of the Lie group. It is the number of continuous real parameters required to describe each operation in the group uniquely.

It is useful at this point to introduce the ideas of compactness and noncompactness. Roughly speaking, a compact space is in some sense finite and a noncompact space is not finite.

Definition: A topological space T is compact if every open cover (set of open sets U_α) has a finite subcover: $\cup_\alpha^{\text{finite}} T \subset U_\alpha$.

In spaces R^n with a Euclidean notion of distance ($|x - x'|^2 = |x_1 - x_1^{'}|^2 + \cdots + |x_n - x_n^{'}|^2$), this definition is equivalent to an older definition of compact spaces: A space is compact if every infinite sequence of points has a subsequence that converges to a point in the space.

Example: In Fig. 2.1 the sphere S^2 is compact and the plane R^2 is not compact. In Fig. 2.2, the circle is compact and the hyperboloid is not compact.

Remark: In R^n every bounded closed subset is compact. 'Closed' means that the set contains all its limit points.

Remark: Compactness is an important topological property because it means that the space is in some sense like a bounded, closed space. For Lie groups it is important because all irreducible representations of compact Lie groups are finite dimensional and can be constructed by rather simple means (tensor product constructions).

2.3 Unification of Algebra and Topology

The rigidity of Lie group structures comes from combining the algebraic and topological properties through smoothness (differentiability) requirements.

Definition: A *Lie group* consists of a manifold M^n that parameterizes the group operations $(g(x), x \in M^n)$ and a combinatorial operation defined by $g(x) \circ g(y) = g(z)$, where the coordinate $z \in M^n$ depends on the coordinates $x \in M^n$ and $y \in M^n$ through a function $z = \phi(x, y)$.

There are two topological axioms for a Lie group.

(i) **Smoothness of the group composition map:** The group composition map $z = \phi(x, y)$, defined by $g(x) \circ g(y) = g(z)$, is differentiable.

(ii) **Smoothness of the group inversion map:** The group inversion map $y = \psi(x)$, defined by $g(x)^{-1} = g(y)$, is differentiable.

It is possible to combine these two axioms into a single axiom, but there is no advantage to this.

Example: For $SL(2;R)$ with parameterization (2.3) the composition function $z = \phi(x, y)$ is constructed easily by matrix multiplication $g(x) \circ g(y) = g(\phi(x, y))$

$$\begin{array}{ccccc} g(x_1, x_2, x_3) & \circ & g(y_1, y_2, y_3) & = & g(z_1, z_2, z_3) \\ \left[\begin{array}{cc} x_1 & x_2 \\ x_3 & \dfrac{1+x_2x_3}{x_1} \end{array}\right] & \times & \left[\begin{array}{cc} y_1 & y_2 \\ y_3 & \dfrac{1+y_2y_3}{y_1} \end{array}\right] & = & \left[\begin{array}{cc} z_1 & z_2 \\ z_3 & \dfrac{1+z_2z_3}{z_1} \end{array}\right] \end{array}$$

where

$$\begin{array}{ccc} g\left(\phi(x_1, x_2, x_3; y_1, y_2, y_3)\right) & = & g(z_1, z_2, z_3) \\ \left[\begin{array}{cc} x_1y_1 + x_2y_3 & x_1y_2 + x_2\,\dfrac{1+y_2y_3}{y_1} \\ x_3y_1 + \dfrac{1+x_2x_3}{x_1}\,y_3 & * \end{array}\right] & = & \left[\begin{array}{cc} z_1 & z_2 \\ z_3 & \dfrac{1+z_2z_3}{z_1} \end{array}\right] \end{array} \tag{2.4}$$

The result is easily read off matrix element by matrix element:

$$\begin{array}{rcl} z_1 = \phi_1(x_1, x_2, x_3; y_1, y_2, y_3) & = & x_1y_1 + x_2y_3 \\ z_2 = \phi_2(x_1, x_2, x_3; y_1, y_2, y_3) & = & x_1y_2 + x_2\,\dfrac{1+y_2y_3}{y_1} \\ z_3 = \phi_3(x_1, x_2, x_3; y_1, y_2, y_3) & = & x_3y_1 + \dfrac{1+x_2x_3}{x_1}\,y_3 \end{array} \tag{2.5}$$

The function ϕ is analytic in its two pairs of arguments provided x_1 and y_1 are bounded away from the x_2-x_3 plane $x_1 = 0$ and the y_2-y_3 plane $y_1 = 0$. In the neighborhood of these values an alternative parameterization of the group is needed.

It is also useful to determine the mapping that takes a group operation into its inverse. We can determine the coordinates (y_1, y_2, y_3) of $[g(x_1, x_2, x_3)]^{-1}$ by setting $(z_1, z_2, z_3) = (1, 0, 0)$ and solving for (y_1, y_2, y_3) in terms of (x_1, x_2, x_3). Or more simply we can compute the inverse of the matrix (2.3)

$$\left[\begin{array}{cc} x_1 & x_2 \\ x_3 & (1+x_2x_3)/x_1 \end{array}\right]^{-1} = \left[\begin{array}{cc} (1+x_2x_3)/x_1 & -x_2 \\ -x_3 & x_1 \end{array}\right] \tag{2.6}$$

The inverse mapping $[g(x)]^{-1} = g(y) = g(\psi(x))$ is

$$\begin{array}{rcccc} \psi_1(x_1, x_2, x_3) & = & y_1 & = & (1+x_2x_3)/x_1 \\ \psi_2(x_1, x_2, x_3) & = & y_2 & = & -x_2 \\ \psi_3(x_1, x_2, x_3) & = & y_3 & = & -x_3 \end{array} \tag{2.7}$$

This mapping is analytic except at $x_1 = 0$, where an alternative parameterization is required. The parameterization shown in Fig. 2.2 handles this problem quite well. Every matrix in $SL(2;R)$ can be written as the product of a symmetric matrix and a rotation matrix, both 2×2 and unimodular. The symmetric matrix is parameterized by a 2-dimensional manifold, the 2-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$. The rotation matrix is parameterized by a point on a circle. Two points $(x, y, |z|, \theta)$ and $(-x, -y, -|z|, \theta + \pi)$ map to the same matrix in $SL(2;R)$. The manifold that parameterizes $SL(2;R)$ is three-dimensional. It is $H^{2+} \times S^1$, where H^{2+} is the upper sheet of the two-sheeted hyperboloid.

2.4 Unexpected Simplification

Almost every Lie group that we will encounter is either a matrix group or else equivalent to a matrix group. This simplifies the description of the algebraic, topological, and continuity properties of these groups. Algebraically, the only group operations that we need to consider are matrix multiplication and matrix inversion. Geometrically, the only manifolds we encounter are those manifolds that can be constructed from matrices by imposing algebraic constraints (*algebraic manifolds*) on the matrix elements. The continuity properties on the matrix elements are simple consequences of matrix multiplication and inversion.

2.5 Conclusion

Lie groups lie at the intersection of the two great divisions of mathematics: algebra and topology. The group elements are points in a manifold, and as such are parameterized by continuous real variables. These points can be combined by an operation that obeys the group axioms. The combinatorial operation $\phi(x, y)$ defined by $g(x) \circ g(y) = g(z) = g(\phi(x, y))$ is differentiable in both sets of variables. In addition, the mapping $y = \psi(x)$ of a group operation to its inverse $[g(x)]^{-1} = g(y) = g(\psi(x))$ is also differentiable.

Unexpectedly, almost all of the Lie groups encountered in applications are matrix groups. This effects an enormous simplification in our study of Lie groups. Almost all of what we would like to learn about Lie groups can be determined by studying matrix groups.

2.6 Problems

1. Construct the analytic mapping $\phi(x,y)$ for the parameterization of $SL(2;R)$ illustrated in Fig. 2.2.

2. Construct the inversion mapping for the parameterization of $SL(2;R)$ given in Fig. 2.2. Show that

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = -\begin{bmatrix} \cos(2\theta) & -\sin(2\theta) & 0 \\ \sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

3. Convince yourself that every matrix M in the group $SL(n;R)$ can be written as the product of an $n\times n$ real symmetric unimodular matrix S and an orthogonal matrix O in $SO(n)$: $M = SO$. Devise an algorithm for constructing these matrices. Show $S = (MM^t)^{1/2}$ and $O = S^{-1}M$. How do you compute the square root of a matrix? Show that O is compact while S and M are not compact.

4. Construct the most general linear transformation $(x,y,z) \to (x',y',z')$ that leaves invariant (unchanged) the quadratic form $z^2 - x^2 - y^2 = 1$. Show that this linear transformation can be expressed in the form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{M_1}} & a \\ & & b \\ \hline a & b & M_2 \end{array}\right] \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$SO(2)$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where the real symmetric matrices M_1 and M_2 satisfy

$$M_1^2 = I_2 + \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 1+a^2 & ab \\ ba & 1+b^2 \end{bmatrix} \quad \text{and}$$

$$M_2^2 = I_1 + \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1+a^2+b^2 \end{bmatrix}$$

5. Construct the group of linear transformations $[SO(1,1)]$ that leaves invariant the quantity $(ct)^2 - x^2$. Compare this with the group of linear transformations $[SO(2)]$ that leaves invariant the radius of the circle $x^2 + y^2$. (This comparison involves mapping trigonometric functions to hyperbolic functions by analytic continuation.)

6. Construct the group of linear transformations that leaves invariant the quantity $(ct)^2 - x^2 - y^2 - z^2$. This is the Lorentz group $O(3,1)$. Four

disconnected manifolds parameterize this group. These contain the four different group operations

$$\begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the $\pm$ signs are incoherent.

7. The group of 2×2 complex matrices with determinant $+1$ is named $SL(2;C)$. Matrices in this group have the structure $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, where $\alpha, \beta, \gamma, \delta$ are complex numbers and $\alpha\delta - \beta\gamma = 1$. Define the matrix X by

$$X = H(x, y, z, ct) = \begin{bmatrix} ct + z & x - iy \\ x + iy & ct - z \end{bmatrix} = ctI_2 + \sigma \cdot \mathbf{x}$$

where $\mathbf{x}$ is the three-vector $\mathbf{x} = (x, y, z)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3) = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli spin matrices.

a. Show that X is hermitian: $X^\dagger \equiv (X^t)^* = X$.
b. Show that the most general 2×2 hermitian matrix can be written in the form used to construct X.
c. If $g \in SL(2;C)$, show that $g^\dagger X g = X' = H(x', y', z', ct')$.
d. How are the new space-time coordinates (x', y', z', ct') related to the original coordinates (x, y, z, ct)? (They are linearly related by coefficients that are bilinear in the matrix elements $\alpha, \beta, \gamma, \delta$ of g and $\alpha^*, \beta^*, \gamma^*, \delta^*$ of its adjoint matrix $g^\dagger$.)
e. Find the subgroup of $SL(2;C)$ that leaves $t' = t$. (It is $SU(2) \subset SL(2;C)$).
f. For any $g \in SL(2;C)$ write $g = kh$, where $h \in SU(2)$, $h^\dagger = h^{-1}$, h has the form $h = EXP(\frac{i}{2}\sigma \cdot \theta)$ and $k \in SL(2;C)/SU(2)$, $k^\dagger = k^{+1}$, k has the form $k = EXP(\frac{1}{2}\sigma \cdot \mathbf{b})$. The three-vector $\mathbf{b}$ is called a boost vector. The three-vectors θ and $\mathbf{b}$ are real. Construct $k^\dagger H(x, y, z, ct) k = H(x', y', z', ct')$. If this is too difficult, choose $\mathbf{b}$ along the z-axis — $\mathbf{b} = (0, 0, b)$.
g. Show that the usual Lorentz transformation law results.
h. Applying $k(b')$ after applying $k(b)$ results in: (a) $k(b' + b)$; (b) two successive Lorentz transformations. Show that the velocity addition law for colinear boosts results.

i. If $\mathbf{b}$ and $\mathbf{b}'$ are not colinear, $k(\mathbf{b}')k(\mathbf{b}) = k(\mathbf{b}'')h(\theta)$. Compute $\mathbf{b}''$, θ. The angle θ is related to the Thomas precession [31].

8. The circumference of the unit circle is mapped into itself under the transformation $\theta \to \theta' = \theta + k + f(\theta)$, where k is a real number, $0 \leq k < 2\pi$, and $f(\theta)$ is periodic: $f(\theta + 2\pi) = f(\theta)$. The mapping must be 1:1, so an additional condition is imposed on $f(\theta)$: $df(\theta)/d\theta > -1$ everywhere. Does this set of transformations form a group? What are the properties of this group?

9. Rational fractional transformations (a, b, c, d) map points on the real line (real projective line RP^1) to the real line as follows:

$$x \to x' = (a, b, c, d)x = \frac{ax + b}{cx + d}$$

The transformations (a, b, c, d) and $(\lambda a, \lambda b, \lambda c, \lambda d) = \lambda(a, b, c, d)$ $(\lambda \neq 0)$ generate identical mappings.

a. Compose two successive rational fractional transformations

$$(A, B, C, D) = (a', b', c', d') \circ (a, b, c, d)$$

and show that the composition is a rational fractional transformation. Compute the values of A, B, C, D.

b. Show that the transformations $(\lambda, 0, 0, \lambda)$ map x to itself.

c. Construct the inverse transformation $x' \to x$, and show that it is $\lambda(d, -b, -c, a)$ provided $\lambda \neq 0$. Such transformations exist if $D = ad - bc \neq 0$.

d. Show that the transformation degeneracy $x' = (a, b, c, d)x = \lambda(a, b, c, d)x$ can be lifted by requiring that the four parameters a, b, c, d describing these transformations satisfy the constraint $D = ad - bc = 1$.

e. It is useful to introduce homogeneous coordinates (y, z) and define the real projective coordinate x as the ratio of these homogeneous coordinates: $x = y/z$. If the homogeneous coordinates transform linearly under $SL(2; R)$ then the real projective coordinates x transform under rational fractional transformations:

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \Rightarrow x' = \frac{y'}{z'} = \frac{a(y/z) + b}{c(y/z) + d} = \frac{ax + b}{cx + d}$$

f. Show that a rational fractional transformation can be constructed that maps three distinct points x_1, x_2, x_3 on the real line to the three standard positions $(0, 1, \infty)$, and that this mapping is

$$x \rightarrow x' = \frac{(x - x_1)(x_2 - x_3)}{(x - x_3)(x_2 - x_1)}$$

What matrix in $SL(2; R)$ describes this mapping? (Careful of the condition $D = 1$.)

g. Use this construction to show that there is a unique mapping of any triple of distinct points (x_1, x_2, x_3) to any other triple of distinct points (x'_1, x'_2, x'_3).

10. The real projective space RP^n is the space of all straight lines through the origin in R^{n+1}. The group $SL(n+1; R)$ maps $x = (x_1, x_2, \ldots, x_{n+1}) \in R^{n+1}$ to $x' \in R^{n+1}$, with $x' \neq 0 \leftrightarrow x \neq 0$ and $x' = 0 \leftrightarrow x = 0$. A straight line through the origin contains $x \neq 0$ and $y \neq 0$ if (and only) $y = \lambda x$ for some real scale factor $\lambda \neq 0$. The scale factor can always be chosen so that y is in the unit sphere in R^{n+1}: $y \in S^n \subset R^{n+1}$. In fact, two values of λ can be chosen: $\lambda = \pm 1/\sqrt{\sum_{i=1}^{n+1} x_i^2}$. In R^3 the straight line containing (x, y, z) can be represented by homogeneous coordinates $(X, Y) = (x/z, y/z)$ if $z \neq 0$. Straight lines through the origin of R^3 are mapped to straight lines in R^3 by $x \rightarrow x' = Mx$, $M \in SL(3; R)$. Show that the homogeneous coordinates representing the two lines containing x and x' are related by the linear fractional transformation

$$\left[\begin{array}{c} X \\ Y \end{array}\right] \rightarrow \left[\begin{array}{c} X' \\ Y' \end{array}\right] =$$

$$\left(\left[\begin{array}{cc} m_{11} & m_{12} \\ m_{21} & m_{22} \end{array}\right]\left[\begin{array}{c} X \\ Y \end{array}\right] + \left[\begin{array}{c} m_{13} \\ m_{23} \end{array}\right]\right) \Big/ \left(\left[\begin{array}{cc} m_{31} & m_{32} \end{array}\right]\left[\begin{array}{c} X \\ Y \end{array}\right] + m_{33}\right)$$

Generalize for linear fractional transformations $RP^n \rightarrow RP^n$.

11. The hyperbolic 2-space $SL(2; R)/SO(2) \simeq \left[\begin{array}{cc} z + x & y \\ y & z - x \end{array}\right]$ consists of the algebraic submanifold in the Minkowski $2 + 1$ dimensional spacetime with metric $(+1, -1, -1)$

$$z^2 - (x^2 + y^2) = 1$$

This submanifold inherits the metric

$$ds^2 = dz^2 - (dx^2 + dy^2)$$

a. Show that

$$-ds^2 = dx^2 + dy^2 - (d\sqrt{1+x^2+y^2})^2 =$$

$$\frac{1}{1+x^2+y^2} \begin{pmatrix} dx & dy \end{pmatrix} \begin{bmatrix} 1+y^2 & -xy \\ -yx & 1+x^2 \end{bmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

b. Introduce polar coordinates $x = r\cos\phi,\ y = r\sin\phi$, and show

$$-ds^2 = \frac{dr^2}{1+r^2} + (rd\phi)^2$$

c. Show that the volume element on this surface is

$$dV = \frac{r\ dr\ d\phi}{\sqrt{1+r^2}}$$

d. Repeat this calculation for $SO(3)/SO(2)$. This space is a sphere $S^2 \subset R^3$: the algebraic manifold in R^3 that satisfies $z^2 + (x^2 + y^2) = 1$ and inherits the metric $ds^2 = dz^2 + (dx^2 + dy^2)$ from this Euclidean space. Show that the metric and measure on S^2 are obtained from the results above for H^2 by the substitutions $1+r^2 \to 1-r^2$. Show that the disk $0 \le r \le 1$, $0 \le \phi \le 2\pi$ maps onto the upper hemisphere of the sphere, with $r = 0$ mapping to the north pole and $r = 1$ mapping to the equator. Show that the geodesic length from the north pole to the equator along the longitude $\phi = 0$ is $s = \int_0^1 dr/\sqrt{1-r^2} = \pi/2$ and the volume of the hemisphere surface is $V = \int_{r=0}^{r=1}\int_{\phi=0}^{\phi=2\pi} dV(r,\phi) = \int_0^1 rdr/\sqrt{1-r^2}\int_0^{2\pi} d\phi = 2\pi$.