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Dimensional Analysis

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We will learn the methods of dimensional analysis through a number of simple examples of slightly increasing interest (read: complexity). These examples will involve the use of the *linalg* package of Maple, which we will use to invert matrices, compute their null subspaces, and reduce matrices to row echelon (Gauss Jordan) canonical form. Dimensional analysis, powerful though it is, is only the tip of the iceberg. It extends in a natural way to ideas of symmetry and general covariance ('what's on one side of an equation is the same as what's on the other').

Example 1a - Period of a Pendulum: Suppose we need to compute the period of a pendulum. The length of the pendulum is l, the mass at the end of the pendulum bob is m, and the earth's gravitational constant is g. The length l is measured in cm, the mass m is measured in gm, and the gravitational constant g is measured in cm/sec^2. The period τ is measured in sec. The only way to get something whose dimensions are sec from m, g, and l is to divide l by g, and take the square root:

$$[\tau] = [\sqrt{l/g}] = [cm/(cm/sec^2)]^{1/2} = [sec^2]^{1/2} = sec \tag{1}$$

In this equation, the symbol $[x]$ is read 'the dimensions of x are $\cdots$'. As a result of this simple (some would say 'sleazy') computation, we realize that the period τ is proportional to $\sqrt{l/g}$ up to some numerical factor which is usually $\mathcal{O}(1)$: in plain English, of order 1. Dimensional analysis is usually accurate to $2^{\pm 1}$, it is *never* off by more than $(2\pi)^{\pm 1}$.

Example 1b - Frequency of Harmonic Oscillator: A mass m is attached to a spring with spring constant k. Find the resonance frequency. Mass m is measured in gm, $[k] = gm/sec^2$. The only way to get something with the dimensions of frequency ($[\nu] = sec^{-1}$) is to construct $\sqrt{k/m}$. Therefore $\nu = \sqrt{k/m}$. If we asked for the angular frequency ω we would get the same result. Since $\omega = 2\pi\nu$, there is a factor 2π difference between the results of the two analyses.

Example 2a - Hydrogen Atom: We would like to estimate the length scale of atomic dimensions 'out of thin air'. The electron, of mass m 'orbits around' the proton and interacts with it through the charge interaction, proportional to e^2. The motion is quantized in terms of Planck's constant $\hbar$. Can we construct something with the dimensions of length from m, e^2, and $\hbar$? Mass is measured in gm. The dimensions of the charge coupling e^2 are determined by recognizing that e^2/r is a (potential) energy, with dimensions $M^1L^2T^{-2}$. We will use capital letters M, L, and T to characterize the three independent dimensional 'directions'. As a result, the charge coupling e^2 has dimensions ML^3T^{-2} and is measured in $gm(cm)^3/sec^2$. The quantum of action $\hbar$ has dimensions $[\hbar] = ML^2T^{-1}$. Since we want to combine these three fundamental quantities in such a way as to construct something with the dimensions of length, we introduce three unknown exponents a, b, and c and write

$$m^a\ (e^2)^b\ \hbar^c = (M)^a\ (ML^3T^{-2})^b\ (ML^2T^{-1})^c = (M)^{a+b+c}\ L^{0a+3b+2c}\ T^{0a-2b-c} \tag{2}$$

and set this result equal to the dimensions of whatever we would like to compute, in this case the Bohr orbit a_0 (characteristic atomic length). This results in a matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{3}$$

We can invert this matrix with the callup *inverse* to find

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & 2 & 3 \end{bmatrix} \tag{4}$$

This allows us to determine the values of the exponents which provide the appropriate combinations of dimensions to construct the characteristic atomic length:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \tag{5}$$

This result tells us that

$$a_0 \sim m^{-1}(e^2)^{-1}(\hbar)^2 = \hbar^2/me^2 \sim 10^{-8}\ cm \tag{6}$$

To construct a characteristic atomic time, we can replace the vector col$[0,1,0]$ in equation (5) by the vector col$[0,0,1]$, giving us the result $\tau_0 \sim \hbar^3/m(e^2)^2$. Finally, to get a characteristic energy, we can form the combination $\mathcal{E} \sim ML^2T^{-2} = m(\hbar^2/me^2)^2(\hbar^3/me^4)^{-2} = me^4/\hbar^2$. Another, and more systematic, way to get this result is to substitute the vector col$[1,2,-2]$ in equation (5).

Example 2b - Hydrogen Atom, Another Way: In the previous example we saw that all the interesting stuff occurred in the powers. Once we set up the products and combined the powers of the various dimensions, we got linear equations which we could solve using Maple (or other). We could jump directly to linear analyses simply by taking logarithms. This brings the powers down to earth, so to speak. The only problem is that we wind up with strange looking creatures like $\log L$: for example, $\log e^2 = \log(ML^3T^{-2}) = 1\log M + 3\log L - 2\log T$. How to interpret these creatures? The easiest way is to stretch the imagination and to consider $\log M$, $\log L$, and $\log T$ as basis vectors in a three dimensional space of (exponents of) dimensions. Using this interpretation we can write the relation between the physical constants m, e^2, $\hbar$ and the dimensions M, L, and T in this logical form

$$\begin{bmatrix} \log m & \log e^2 & \log \hbar \end{bmatrix} = \begin{bmatrix} \log M & \log L & \log T \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -2 & -1 \end{bmatrix} \tag{7}$$

The inversion works as before, and we find

$$\begin{bmatrix} \log M & \log L & \log T \end{bmatrix} = \begin{bmatrix} \log m & \log e^2 & \log \hbar \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & 2 & 3 \end{bmatrix} \tag{8}$$

To determine characteristic length, time, and energy scales, simply multiply on the right by the column vectors col$[0,1,0]$, col$[0,0,1]$, and col$[1,2,-2]$.

Example 3a - Hydrogen Atom: A More Sophisticated Approach: While the method described above is effective, it involves computing a matrix inverse. Here is another method which is even easier. We will compute the energy $\mathcal{E}$ for the hydrogen atom by adjoining $\mathcal{E}$ to the list of fundamental constants, and then trying to construct a dimensionless combination of constants involving $\mathcal{E}, m, e^2$, and $\hbar$. We have

$$\left[\begin{array}{cccc} \log\mathcal{E} & \log m & \log e^2 & \log\hbar \end{array}\right] = \left[\begin{array}{ccc} \log M & \log L & \log T \end{array}\right] \left[\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 2 & 0 & 3 & 2 \\ -2 & 0 & -2 & -1 \end{array}\right] \tag{9}$$

Next, we look for a *null vector* of this matrix. This is easily found using the call *nullspace*. The result is col$[-1, 1, 2, -2]$. The 3×4 matrix above annihilates this vector. This means

$$\left[\begin{array}{cccc} \log\mathcal{E} & \log m & \log e^2 & \log\hbar \end{array}\right] \left[\begin{array}{r} -1 \\ 1 \\ 2 \\ -2 \end{array}\right] = 0 \tag{10}$$

In short, $\mathcal{E}^{-1}m^1(e^2)^2(\hbar)^{-2} \sim 1$, and as before $\mathcal{E} \simeq me^4/\hbar^2$.

Example 3b - Hydrogen Masterpiece - with all the interesting quantities: We now set up the dimensional analysis computation in its final elegant formulation. We do this by computing the characteristic Bohr size (a_0), energy ($\mathcal{E}$), and time (τ) scales for the nonrelativistic hydrogen atom. We do this by adjoining the formal logs of these guys to our list of nonrelativistic fundamental constants for the hydrogen atom. In the relations which follow we omit the 'log' expressions: including them caused the equation to run off the edge of the page.

$$\left[\begin{array}{cccccc} a_0 & \mathcal{E} & \tau & m & e^2 & \hbar \end{array}\right] = \left[\begin{array}{ccc} M & L & T \end{array}\right] \left[\begin{array}{rrrrrr} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & -2 & 1 & 0 & -2 & -1 \end{array}\right] \tag{11}$$

Next, we ask for the null vectors of the 3×6 matrix. There are three: col$[-1, -1, 0, 0, 1, 0]$, col$[2, -1, -2, 1, 0, 0]$, and col$[0, -1, -1, 0, 0, 1]$. Each null vector defines a different dimensionless product. These three dimensionless products yield $a_0\mathcal{E} = e^2$, $ma_0^2 = \mathcal{E}\tau^2$, and $\mathcal{E}\tau = \hbar$. However, what we would really like is to have each of the dimensionless products involve only one of the three quantities a_0, $\mathcal{E}$, and τ. This is possible if the three basis vectors in the null space each involves only one of the three physical quantities $\mathcal{E}$, a_0, and τ. In fact, even more preferable would be if the coefficient of each of these would be $+1$. In other words, it would be nice to have the basis vectors in the null space chosen in row echelon form. The callup to *gaussjord* does precisely this.

Here is how we implement this. Basis vectors for the nullspace are computed by the call to *nullspace*. The three basis vectors in the output are stacked into a 3×6 matrix

$$B5 = \left[\begin{array}{rrrrrr} -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \tag{12}$$

The callup *gaussjord*$(B5)$ returns the desired basis vectors

$$\left[\begin{array}{rrrrrr} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 2 & -3 \end{array}\right] \tag{13}$$

From the first row we learn $a_0 m^1 (e^2)^1 \hbar^{-2}$ is dimensionless, so that $a_0 = \hbar^2/me^2$. In the same way, from the second row we learn $\mathcal{E} = me^4/\hbar^2$, and $\tau = \hbar^3/me^4$.

Example 4 - Classical Electrodynamics: Quantum theories involve the fundamental constants m, e^2, and $\hbar$. Classical theories do not involve the quantum of action $\hbar$, but relativistic classical theories (Classical Electrodynamics) do involve the speed of light c, with $[c] = LT^{-1}$. We can repeat the calculations above, but on the classical level. We find

$$\left[\begin{array}{cccccc} r_0 & \mathcal{E} & \tau & m & e^2 & c \end{array} \right] = \left[\begin{array}{ccc} M & L & T \end{array} \right] \left[\begin{array}{rrrrrr} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 3 & 1 \\ 0 & -2 & 1 & 0 & -2 & -1 \end{array} \right] \tag{14}$$

The three null vectors are computed as before.

$$\left[\begin{array}{rrrrrr} -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & -2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \tag{15}$$

The row echelon form is also computed as before. As a last modification of this machine for computing appropriate products of fundamental coupling constants, we multiply the row echelon matrix by -1 using the callup *scalarmul*(“$, -1$). Then the powers of the coupling constants are simply read off the resulting matrix: In reduced row echelon form, null basis vectors are

$$\left[\begin{array}{rrrrrr} -1 & 0 & 0 & -1 & 1 & -2 \\ 0 & -1 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 & 1 & -3 \end{array} \right] \tag{16}$$

From the first row we see directly that $a_0 = m^{-1}(e^2)^1 c^{-2} = e^2/mc^2$. This is the classical radius of the electron, which is obtained by equating the energy stored in its electromagnetic field, e^2/r_0, with its rest energy, mc^2. From the second row we learn $\mathcal{E} = mc^2$. From the third row we find $\tau = e^2/mc^3$. This is the time it takes light, moving at c, to go a distance equal to the classical radius of the electron e^2/mc^2.

Example 5 - Quantum Gravity and Planck Scales: Any theory which involves gravitation, quantum mechanics, and relativity must deal with the three fundamental constants G, $\hbar$, and c. We ask: What are the fundamental units of length, time, energy, and mass in such a theory. These are called the *Planck scales.* By now the dimensional analysis program has been reduced to this boring algorithm.

First, set up the formal matrix equation relating the logs of the physical units to the logs of the dimensions

$$\left[\begin{array}{ccccccc} m & a_0 & \tau & \mathcal{E} & G & \hbar & c \end{array} \right] = \left[\begin{array}{ccc} M & L & T \end{array} \right] \left[\begin{array}{rrrrrrr} 1 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & -2 & -2 & -1 & -1 \end{array} \right] \tag{17}$$

The four null vectors are

$$\left[\begin{array}{rrrrrrr} -1 & -2 & 2 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -3 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \tag{18}$$

The row echelon form is

$$
\left[\begin{array}{rrrrrrr}
-1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\
0 & 0 & -1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \\
0 & 0 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2}
\end{array}\right] \tag{19}
$$

As a result, we read off directly from this row echelon matrix

$$
\begin{array}{rcl}
m & \simeq & \sqrt{\hbar c/G} \\
a_0 & \simeq & \sqrt{G\hbar/c^3} \\
\tau & \simeq & \sqrt{G\hbar/c^5} \\
\mathcal{E} & \simeq & \sqrt{\hbar c^5/G}
\end{array} \tag{20}
$$

Example 6 - Fundamental Dimensionless Ratios: So far we have concentrated on constructing physical quantities $(a_0, \mathcal{E}, \tau)$ from the fundamental constants. We have introduced four: $e^2, \hbar, c, G$ and chosen various subsets of them to describe quantum theories $(m, e^2, \hbar)$, classical theories (m, e^2, c), and quantum gravity $(G, \hbar, c)$. Since there are four fundamental constants but the dimension space is only three dimensional, with basis vectors $\log M, \log L, \log T$, then the four interaction constants cannot all be independent. There must be some linear relation among their exponents. Put another way, there must be some product of the four which is dimensionless. We compute this product now.

$$
\left[\begin{array}{cccc} e^2 & \hbar & c & G \end{array}\right] = \left[\begin{array}{ccc} M & L & T \end{array}\right] \left[\begin{array}{rrrr}
1 & 1 & 0 & -1 \\
3 & 2 & 1 & 3 \\
-2 & -1 & -1 & -2
\end{array}\right] \tag{21}
$$

The null space is one dimensional, spanned by col$[-1, 1, 1, 0]$. This means that the product $(e^2)^{-1}\hbar c = 1/\alpha$ is dimensionless. The ratio $\alpha = e^2/\hbar c$ is called the *fine structure constant.* It is the most accurately measured of all the known dimensionless physical constants. Its value is $\alpha = 1/137.03608\ldots$. Since α is small, $\alpha \simeq 0.007..$, it is very useful to make perturbation expansions in powers of α. They converge 'quickly'. This is why Quantum Electrodynamics has been so successful in predicting the behavior of charged particles interacting with radiation.

Summary of Results: In Table 1 we summarize the expressions for important physical quantities, mass (m), length (a_0), time (τ), momentum (p), and energy $(\mathcal{E})$, as well as approximate numerical values, for the three different physical theories treated above.

Summary of Algorithm: Introduce the k_1 fundamental units for the problem (*e.g.*, M, L, T). Introduce the k_3 constants which define the physics (*e.g.*, $G, e^2, \hbar, c$). List the k_2 variables which are to be estimated (*e.g.*, $m, a_0, \tau, p, \mathcal{E}$). Create a $k_1 \times (k_2 + k_3)$ matrix in which each column describes the powers of the corresponding variable or physical constant in terms of the k_1 fundamental dimensions. The matrix has the structure

	m	a_0	τ	p	$\mathcal{E}$	G	e^2	$\hbar$	c
M				1	1				
L				1	2				
T				−1	−2				

(22)

Table 1: Expressions and numerical values for mass, length, time, momentum, and energy scales in Nonrelativistic Quantum Mechanics, Classical Electrodynamics, and Quantum Gravity, as determined from dimensional analysis. Inputs: $m_e = 0.911 \times 10^{-27} gm$, $e^2 = 2.308 \times 10^{-19} gm\ cm^3/sec^2$, $\hbar = 1.055 \times 10^{-27} gm\ cm^2/sec$, $c = 2.9929 \times 10^{10} cm/sec$, $G = 6.672 \times 10^{-8} cm^3/gm\ sec^2$.

Symbol Units	Non − Rel. Q.M. $m, e^2, \hbar$	Rel. Q.M $m, \hbar, c$	Classical E.D. m, e^2, c	Quantum Gravity $G, \hbar, c$	Gravo− Dynamics G, e^2, c
m	m	m	m	$\sqrt{\hbar c/G}$	$\sqrt{e^2/G}$
a_0	$\hbar^2/me^2$	$\hbar/mc$	e^2/mc^2	$\sqrt{G\hbar/c^3}$	$\sqrt{Ge^2}/c^2$
τ	$\hbar^3/me^4$	$\hbar/mc^2$	e^2/mc^3	$\sqrt{G\hbar/c^5}$	$\sqrt{Ge^2}/c^3$
p	$me^2/\hbar$	mc	mc	$\sqrt{\hbar c^3/G}$	$\sqrt{e^2/Gc}$
$\mathcal{E}$	$me^4/\hbar^2$	mc^2	mc^2	$\sqrt{\hbar c^5/G}$	$\sqrt{e^2/Gc^2}$
$[m] = gm$	$9.1 - 28$	$9.1 - 28$	$9.1 - 28$	$2.8 - 05$	$1.9 - 06$
$[a_0] = cm$	$5.3 - 09$	$3.9 - 11$	$2.8 - 13$	$1.6 - 33$	$1.4 - 34$
$[\tau] = sec$	$2.4 - 17$	$1.3 - 21$	$9.4 - 24$	$5.3 - 44$	$4.2 - 45$
$[p] = gm\ cm/sec$	$2.0 - 19$	$2.7 - 17$	$2.7 - 17$	$6.5 + 05$	$5.6 + 04$
$[\mathcal{E}] = gm\ cm^2/sec^2$	$4.4 - 11$	$8.2 - 07$	$8.2 - 07$	$2.0 + 16$	$1.7 + 15$

Compute the nullspace of this matrix. This is done with the callup *nullspace*. Every vector in this space defines a dimensionless product. Put the null basis into canonical row echelon form. This is done with the callup *gaussjord*. Then multiply this matrix by -1 using the callup $scalarmul(``, -1)$ Then, if the rank of the $k_1 \times k_3$ submatrix is maximal ($= k_1$), each of the first k_2 row echelon null basis vectors involves only one of the physical variables from the subset with k_2 entries, as well as some or all the physical constants from the subset with k_3 elements. Exponents for the powers of the k_3 coupling constants in the expressions for the k_2 physical variables are then simply read from the resulting matrix. Any remaining null basis vectors describe dimensionless products of the k_3 input physical constants.

Example 7 - Stefan-Boltzmann Constant σ: Shortly after Maxwell proposed the equations of electrodynamics, Boltzmann studied a Carnot engine in which the working fluid was not a gas, but electromagnetic radiation at thermodynamic equilibrium at temperature T. He showed that the energy density of this radiation is proportional to the fourth power of the temperature: $u(T) \sim T^4$ (erg/cm^3). From there it was a short step to show that black bodies radiated energy at a rate σT^4 per unit time per unit area. The constant σ had been experimentally determined by Stefan. Estimate σ.

This problem involves radiation (c), quantum mechanics ($\hbar$), and statistical mechanics ($k = 1.38066 \times 10^{-16}$ $erg/^oK$) (K = degree Kelvin). The dimensions of the Stefan-Boltzmann constant σ are determined from $[\sigma T^4] = ML^2T^{-2}/(T \times L^2)$, so that $[\sigma] = M^1T^{-3}K^{-4}$. The dimension matrix is

$$
\begin{array}{c|cccc}
 & \sigma & c & \hbar & k \\
\hline
M & 1 & 0 & 1 & 1 \\
L & 0 & 1 & 2 & 2 \\
T & -3 & -1 & -1 & -2 \\
K & -4 & 0 & 0 & -1
\end{array}
\tag{23}
$$

The null vector is col$[-1, -2, -3, +4]$, so that $\sigma \simeq k^4/c^2\hbar^3 = 3.4 \times 10^{-5} erg\ cm^{-2}\ sec^{-1}\ K^{-4}$. In fact,

$$
\sigma = \frac{\pi^4}{60}\frac{k^4}{c^2\hbar^3} = 5.670 \times 10^{-5}\frac{ergs}{cm^2\ sec\ K^4}
$$

A lot of exciting physics goes into the computation of the numerical factor $\pi^4/60 = 1.6235$.

Example 8 - Water Waves: A surface typically separates two fluids which have different densities, such as air and water. The propagation of waves along the interface between the two fluids is governed by some sort of (complicated) wave equation. Wave equations always have the structure

$$
(\text{Differential Operator in } x)\, \psi(x,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(x,t) = \text{Forcing Term}
$$

Here v is the phase velocity. If ω is the angular frequency ($\omega = 2\pi\nu$) and k is the wave number ($k = 2\pi/\lambda$, λ = wavelength), then $\lambda\nu = v$. A *dispersion relation* relates ω with k. Dispersion relations take the form $\omega^2 = \omega^2(k)$. A dispersion relation can be obtained by the usual straategem of writing $\psi(x,t) \sim \phi(x)e^{i\omega t}$ and substitution into a wave equation. Then the dispersion relation assumes the form $\omega^2(k) = v^2 g(k)$ or more simply $\omega^2(k) = f(k)$.

At an air-fluid interface, we consider the competition between waves in which the dominant restoring force is gravity and those for which the dominant restoring force is surface tension. The important parameters are

Angular Frequency	$\omega = 2\pi\nu$	T^{-1}
Wave Number	$k = 2\pi/\lambda$	L^{-1}
Gravitational Acceleration	g	LT^{-2}
Density	ρ	ML^{-3}
Surface Tension	σ	MT^{-2}
Kinematic Surface Tension	σ/ρ	L^3T^{-2}
Viscosity	μ	$ML^{-1}T^{-1}$
Kinematic Viscosity	$\nu = \mu/\rho$	L^2T^{-1}

In the gravity wave limit the acceleration is independent of the density. The only parameters which are important are ω, k, and g. In this limit $\omega^2 \sim gk$. In the surface tension dominated limit g does not play a role. The dispersion relation involves ω^2, k, σ, and ρ. We have uniquely $\omega^2 \sim (\sigma/\rho)k^3$. The general dispersion relation has the form

$$
\omega^2 = gk + \frac{\sigma}{\rho}k^3
$$

The wave speed is $v = \omega/k = \lambda\nu = \sqrt{(\sigma/\rho)k + g/k}$. The crossover wave number, separating gravity from surface waves, is obtained from $gk = (\sigma/\rho)k^3$, or $k_c = \sqrt{g\rho/\sigma}$.

Example 9 - Surface Tension and Viscosity: If the amplitude of a surface wave becomes too large, the waves break and droplets are ejected. The breakup limits are studied experimentally by placing the fluid in a pan and oscillating the pan vertically with an amplitude z and oscillation frequency ω_0, and consequently a surface acceleration $a = z\omega_0^2$. The amplitude and frequency at which breakup occurs depends on the surface tension σ and the kinematic viscosity ν. We investigate the two limits using dimensional analysis. We compute the acceleration of the surface at which breakup occurs in the two limits.

In the surface tension dominated case breakup occurs when the amplitude h of the wave is comparable to the wavelength λ. Thus, $h \sim z = a/\omega_0^2$ is comparable to $\lambda = \left((\sigma/\rho)\omega_0^{-2}\right)^{1/3}$, so that

$$a = c_1\omega_0(\omega_0\sigma/\rho)^{1/3}$$

In the viscosity dominated regime, breakup occurs when the power input is equal to the viscous dissipation. Both have dimension L^2T^{-3} (per unit mass). The power input depends only on the oscillation amplitude z (or equivalently, on the acceleration a) and frequency ω_0. The power input is a^2/ω_0. The viscous dissipation is proportional to the kinematic viscosity ν, the velocity of the surface $z\omega_0 = a/\omega_0$, and the wavelength λ of the oscillation, so that the energy dissipated per unit mass is $\nu(a/(\omega_0\lambda))^2$. As a result

$$a = c_2\omega_0(\omega_0\nu)^{1/2}$$

with $c_2 = (z/\lambda)$.

Experiments have determined the values of c_1 and c_2 (Christopher L. Goodrich, W. Tao Shi, H. G. E. Hentschel, and Daniel P. Lathrop, Viscous effects in droplet-ejecting capillary waves, Phys. Rev. E**51**, 472-475 (1997)):

$$\begin{aligned} c_1 &= 0.261 \qquad \text{surface tension} \\ c_2 &= 1.306 \qquad \text{viscosity} \end{aligned}$$